Chapter 6

Inverses, Logs, and Exponentials

One-to-One Functions

Definition 1. *A function f is* one-to-one *if it never takes the same value twice. That is,*

$$f(x_1) \neq f(x_2) \quad \textit{whenever } x_1 \neq x_2.$$

Examples. Which function is one-to-one?

Example 1. Decide whether $y = x$ is one-to-one.

Solution:

Example 2. Decide whether $y = x^2$ is one-to-one.

Solution:

Horizontal Line Test

Theorem 2. *A function f is one-to-one if and only if its graph $y = f(x)$ passes the* Horizontal Line Test*, meaning every horizontal line intersects the graph at most once.*

Using Derivatives to Test One-to-One

Theorem 3. *A function whose derivative is either always positive or always negative on an interval is one-to-one on that interval.*

Example 3. Show that $g(x) = \sqrt{4x+4}$ is one-to-one on its domain.

Solution:

Inverse Functions

Definition 4. *If f is one-to-one, we define an inverse function f^{-1} by*

$$f^{-1}(y) = x \quad \textit{if and only if} \quad f(x) = y.$$

Example 4. If $f(x) = x^3 + 1$, find $f^{-1}(9)$.

Solution:

Note.

$$\text{Domain of } f^{-1} = \text{Range of } f, \qquad \text{Range of } f^{-1} = \text{Domain of } f.$$

Finding a Formula for f^{-1}

To find a formula for f^{-1}:

1. Write $y = f(x)$.
2. Switch x and y everywhere.
3. Solve for y to get $y = f^{-1}(x)$.

Example 5. Find a formula for the inverse of $g(x) = \sqrt{4x+4}$.

Solution:

Composing f and f^{-1}

Theorem 5. *If f is one-to-one, then*

$$f^{-1}(f(x)) = x \qquad \textit{and} \qquad f(f^{-1}(x)) = x.$$

Example 6. Let $g(x) = \sqrt{4x+4}$ and $g^{-1}(x) = \frac{x^2-4}{4}$. Verify $g^{-1}(g(x)) = x$.

Solution:

Example 7. Verify $g(g^{-1}(x)) = x$ (be careful about the domain).

Solution:

Graphs of Inverse Functions

Theorem 6. *The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ across the line $y = x$.*

Calculus with Inverse Functions

Theorem 7. *Recall $x = f(f^{-1}(x))$. Using implicit differentiation and the chain rule:*

$$1 = f'(f^{-1}(x)) \cdot (f^{-1})'(x) \quad \Rightarrow \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example 8. Find $(f^{-1})'(-\pi/2)$ if $f(x) = \frac{x}{2} + \sin(x) - \frac{\pi}{2}$.

Solution:

Example 9. Find $(f^{-1})'(65)$ where $f(x) = x^3 + 1$.

Solution:

Example 10. If f is one-to-one and differentiable and $f^{-1}(4) = 6$ and $f'(6) = \pi$, find $(f^{-1})'(4)$.

Solution:

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SECTION 6.2: THE NATURAL LOGARITHM

In Calculus I, you learned that

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1).$$

What happens if $n = -1$?

Definition of the Natural Logarithm

Definition 1. *We define the natural logarithm by*

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \qquad (x > 0).$$

By construction, $\ln x$ is an antiderivative of $\frac{1}{x}$.

Note. $\ln x$ represents the area under $y = \frac{1}{t}$ from $t = 1$ to $t = x$ for $x > 1$, and the negative of the area from $t = x$ to $t = 1$ for $0 < x < 1$.

Examples

$$\ln(2) = \int_1^2 \frac{1}{t}\,dt, \qquad \ln\left(\frac{1}{2}\right) = -\int_{1/2}^1 \frac{1}{t}\,dt.$$

If we want the actual value of $\ln(2)$, we can approximate it using a Riemann sum.

Properties of $\ln x$

1. Domain: $(0,\infty)$
2. Range: $(-\infty,\infty)$
3. $$\ln x > 0 \text{ if } x > 1, \quad \ln x = 0 \text{ if } x = 1, \quad \ln x < 0 \text{ if } 0 < x < 1$$
4. $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
5. The graph of $y = \ln x$ is increasing, continuous, and concave down on $(0,\infty)$.
6. $f(x) = \ln x$ is one-to-one.
7. There exists a unique number e such that $\ln e = 1$.

Algebraic Properties of $\ln x$

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
4. $\ln(a^r) = r \ln a$

Examples

Example 1. Expand

$$\ln\left(\frac{x^2\sqrt{x^2+1}}{x^3}\right).$$

Solution:

Example 2. Simplify

$$\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+1).$$

Solution:

Example 3. Evaluate

$$\int_1^{e^2} \frac{1}{t}\,dt.$$

Solution:

Limits Involving $\ln x$

$$\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

Extending the Logarithm

$$\ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

Then

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \qquad \int \frac{1}{x}\,dx = \ln|x| + C.$$

More generally,

$$\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}, \qquad \int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C.$$

Example 4. Find

$$\frac{d}{dx}\left(\ln \sqrt[3]{x-1}\right).$$

Solution:

Example 5. Evaluate

$$\int \frac{x}{3-x^2}\,dx.$$

Solution:

Logarithmic Differentiation

To differentiate $y = f(x)$, it is sometimes easier to use logarithmic differentiation:

1. Take ln of both sides and simplify.
2. Differentiate with respect to x.
3. Solve for $\frac{dy}{dx}$.

Example 6. Find the derivative of

$$y = \sqrt[4]{\frac{x^2+1}{x^2-1}}.$$

Solution:

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SECTION 6.3: THE NATURAL EXPONENTIAL FUNCTION

Last Time: $\ln(x)$

- Properties: one-to-one; domain $(0,\infty)$, range $(-\infty,\infty)$.
- $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ and $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.
- Logarithmic differentiation.

Since $f(x) = \ln(x)$ is one-to-one, it has an inverse function $f^{-1}(x)$, often written $\exp(x)$, defined by

$$y = \ln(x) \quad \Longleftrightarrow \quad x = \exp(y).$$

We also write $\exp(x) = e^x$.

A Helpful Chart

$y = \ln(x)$	$x = \exp(y)$
$0 = \ln(1)$	$1 = \exp(0)$
$1 = \ln(e)$	$e = \exp(1)$
$2 = \ln(e^2)$	$e^2 = \exp(2)$
$r = \ln(e^r)$	$e^r = \exp(r)$

Thus,

$$y = \ln(x) \Longleftrightarrow e^y = x, \qquad e^{\ln(x)} = x, \qquad \ln(e^x) = x.$$

Example 1. Solve for x if $\ln(2x+1) = 3$.

Solution:

Example 2. Solve for x if $e^{(x+1)/3} = 7$.

Solution:

Limits

From the graph (and properties),

$$\lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to\infty} e^x = \infty.$$

Example 3. Compute $\lim_{x\to\infty} \dfrac{e^x}{2e^x - 1}$.

Solution:

Rules of Exponents

From algebraic properties of ln, we get:

$$e^{x+y} = e^x e^y, \qquad e^{x-y} = \frac{e^x}{e^y}, \qquad \left(e^x\right)^y = e^{xy}.$$

Example 4. Simplify $\dfrac{e^{x^2} e^{2x}}{(e^{x+1})^2}$.

Solution:

Derivative and Chain Rule

Using logarithmic differentiation:

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x).$$

Integrals:

$$\int e^x\,dx = e^x + C, \qquad \int g'(x)e^{g(x)}\,dx = e^{g(x)} + C.$$

Example 5. Compute $\frac{d}{dx}\Big(\sin(e^{x^2+1})\Big)$.

Solution:

Example 6. Compute $\int \sec^2(x)\, e^{\tan x}\, dx$.

Solution:

Old Exam Questions (from notes)

Example 7. The function $f(x) = \frac{x^3}{3} + 3x + 2e^x$ is one-to-one. Compute $f^{-1\prime}(1)$.

Solution:

Example 8. Compute $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^x}\,dx$.

Solution:

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SECTION 6.4: GENERAL EXPONENTIALS AND LOGARITHMS

Reminder

For $a > 0$ and rational r,

$$e^{r\ln(a)} = e^{\ln(a^r)} = a^r.$$

Definition 1. *Let $a > 0$. For any real number x, define the exponential function with base a by*

$$a^x = e^{x\ln(a)}.$$

Rules of Exponents

$$a^{x+y} = a^x a^y, \qquad a^{x-y} = \frac{a^x}{a^y}, \qquad (a^x)^y = a^{xy}, \qquad (ab)^x = a^x b^x.$$

Differentiation and Integration

Differentiate $a^x = e^{x\ln(a)}$:

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

Integrate:

$$\int a^x\,dx = \int e^{x\ln(a)}\,dx = \frac{1}{\ln(a)}a^x + C.$$

Example 1. Compute $\frac{d}{dx}\left(x^5 + 5^x\right)$.

Solution:

Example 2. Compute $\int 2x \cdot 2^{x^2}\,dx$.

Solution:

General Logarithms

Since $f(x) = a^x$ is one-to-one for $a \neq 1$, it has an inverse function called $\log_a(x)$:

$$y = \log_a(x) \quad \Longleftrightarrow \quad a^y = x.$$

In particular, $\log_e(x) = \ln(x)$.

Change of Base

Starting from $y = \log_a(x) \iff a^y = x$, take ln of both sides:

$$\ln(a^y) = \ln(x) \iff y\ln(a) = \ln(x) \iff \boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}} \quad (a > 0,\ a \neq 1).$$

Properties of $\log_a(x)$

$$\log_a(1) = 0, \qquad \log_a(xy) = \log_a(x) + \log_a(y), \qquad \log_a(x^r) = r\log_a(x)$$

(when the expressions make sense; same-base rule).

Derivative

$$\frac{d}{dx}\big(\log_a(x)\big) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x\ln(a)}.$$

Example 3. Compute $\frac{d}{dx}\big(\log_{10}(2+\sin x)\big)$.

Solution:

e as a Limit

Since $f(x) = \ln(x)$ has $f'(1) = 1$:

$$1 = f'(1) = \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\ln\big((1+x)^{1/x}\big).$$

Apply e^x to both sides:

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Let $n = \frac{1}{x}$ so $n \to \infty$:

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}.$$

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SECTION 6.5: EXPONENTIAL GROWTH AND DECAY

Many quantities grow or decay at a rate proportional to their size (population, radioactive substances, bacteria, etc.). Mathematically:

$$\frac{dy}{dt} = ky.$$

If $k > 0$ this is *natural growth*; if $k < 0$ this is *natural decay*.

General Solution

$$y = Ce^{kt}.$$

If $y(0) = y_0$, then $C = y_0$, so

$$\boxed{y(t) = y_0 e^{kt}}.$$

Population Growth Example (Rabbits)

In 1971, 8 rabbits were released on an island in Japan. In 2011, there were 300 rabbits. Let t be years since 1971. Then $P(0) = 8$ and $P(40) = 300$.

Example 1. Find k and a formula for $P(t)$.

Solution:

Example 2. Estimate the rabbit population in 2023.

Solution:

Radioactive Decay and Half-Life

The *half-life* is the time required for half of the substance to decay.

Example 3. Plutonium-233 has a half-life of 20 minutes. If $m(0) = 100$ mg, find $m(t)$.

Solution:

Example 4. How much remains after one hour?

Solution:

Example 5. How long until 10 mg remains?

Solution:

Compound Interest

If $A(0)$ is invested at rate r, compounded n times per year, then after t years:

$$\boxed{A(t) = A(0)\left(1+\frac{r}{n}\right)^{nt}}.$$

Continuously Compounded Interest

$$\lim_{n\to\infty} A(0)\left(1+\frac{r}{n}\right)^{nt} = A(0)e^{rt}.$$

So

$$\boxed{A(t) = A(0)e^{rt}}.$$

Example 6. If I borrow $50,000 at a 10% interest rate, compounded quarterly, how much will I owe after 5 years?

Solution:

Example 7. If the same loan is compounded continuously, how much after 5 years?

Solution:

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SECTION 6.6: INVERSE TRIGONOMETRIC FUNCTIONS

Today

Define inverse functions for $\sin(x)$, $\cos(x)$, and $\tan(x)$.

The Function $\sin^{-1} x$ / $\arcsin(x)$

Since $y = \sin x$ is not one-to-one, restrict its domain to $[-\pi/2, \pi/2]$. The inverse is $\sin^{-1} x$ or $\arcsin(x)$.

Caution: $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$, not $\sin^{-1} x$.

Properties

$$\text{domain}(\arcsin x) = [-1, 1], \qquad \text{range}(\arcsin x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

$$\sin(\arcsin x) = x \text{ for } x \in [-1, 1], \qquad \arcsin(\sin x) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Example 1. Compute $\arcsin\left(\frac{1}{\sqrt{2}}\right)$.

Solution:

Example 2. Compute $\arcsin(\sin \pi)$.

Solution:

Example 3. Find a formula for $\cos(\arcsin x)$.

Solution:

Differentiate $\sin y = x$:

$$\cos y \cdot \frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

So

$$\boxed{\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}}.$$

Example 4. Differentiate $f(x) = \arcsin(e^x)$. What is the domain of f?

Solution:

The Function $\cos^{-1} x$ / $\arccos(x)$

Restrict $y = \cos x$ to $[0, \pi]$. Then $\arccos x$ has:

$$\text{domain} = [-1, 1], \qquad \text{range} = [0, \pi].$$

$$\cos(\arccos x) = x, \qquad \arccos(\cos x) = x \text{ for } x \in [0, \pi].$$

Differentiate $\cos y = x$:

$$-\sin y \cdot \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

So

$$\boxed{\frac{d}{dx}\left(\arccos x\right) = -\frac{1}{\sqrt{1-x^2}}}.$$

The Function $\tan^{-1} x$ / $\arctan(x)$

Restrict $y = \tan x$ to $(-\pi/2, \pi/2)$. Then:

$$\text{domain}(\arctan x) = (-\infty, \infty), \qquad \text{range}(\arctan x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Differentiate $\tan y = x$:

$$\sec^2(y)\frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y) = \frac{1}{1+x^2}.$$

So

$$\boxed{\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}}.$$

Example 5. Evaluate $\sin(\arctan(1/\sqrt{3}))$.

Solution:

Integration Formulas

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

Example 6. Compute $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$.

Solution:

Example 7. Compute $\int \frac{1}{x^2 + a^2}\, dx$ (for $a > 0$).

Solution:

Last Time

- $\arcsin x$, $\arccos x$, $\arctan x$
- Integration formulas:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

Indeterminate Forms

Definition 1. *An indeterminate form of type* $\frac{0}{0}$ *is the limit of a quotient* $\frac{f(x)}{g(x)}$ *where* $f(x) \to 0$ *and* $g(x) \to 0$.

Definition 2. *An indeterminate form of type* $\frac{\infty}{\infty}$ *is the limit of a quotient* $\frac{f(x)}{g(x)}$ *where* $f(x) \to \pm\infty$ *and* $g(x) \to \pm\infty$.

Motivating Idea: "Which function wins?"

Example from notes: $\lim_{x\to\infty} \frac{x}{e^x}$. Since e^x grows faster than x, the limit is 0.

L'Hospital's Rule

Theorem 3 (L'Hospital's Rule)**.** *Assume* f *and* g *are differentiable and* $g'(x) \neq 0$ *near the point of interest. If* $\frac{f(x)}{g(x)}$ *is an indeterminate form of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$*, then*

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$*).*

Example 1. Use L'Hospital's Rule to compute $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Solution:

Type $0 \cdot \infty$

Definition 4. *An indeterminate form of type* $0 \cdot \infty$ *is the limit of a product* $f(x)g(x)$ *where* $f(x) \to 0$ *and* $g(x) \to \pm\infty$.

Example 2. Compute $\lim_{x \to 0^+} x \ln x$.

Solution:

Type $\infty - \infty$

Definition 5. *An indeterminate form of type $\infty - \infty$ is the limit of a difference $f(x) - g(x)$ where both $f(x) \to \infty$ and $g(x) \to \infty$.*

Example 3. Compute $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$.

Solution:

Indeterminate Powers

Definition 6. *An indeterminate power is the limit of $f(x)^{g(x)}$ where:*

$$0^0, \qquad \infty^0, \qquad 1^\infty$$

occur.

Example 4. Compute $\lim_{x \to 0^+} x^x$.

Solution:

Method for Indeterminate Powers

1. Let $L = \lim f(x)^{g(x)}$.
2. Compute $\ln(L) = \lim g(x) \ln(f(x))$ (often using L'Hospital).
3. Then $L = e^{\ln(L f(L)}$.

Chapter 7
Integration Techniques

Integration by Parts

Start with the Product Rule:

$$(uv)' = u'v + uv'$$

Integrate both sides:

$$\int u\,dv = uv - \int v\,du$$

This is Integration by Parts.

LIATE (typical choice guide): Logarithmic, Inverse trig, Algebraic, Trig, Exponential.

Example 1.

Compute $\int xe^x\,dx$.

Solution:

Example 2.

Compute $\int x \cos x \, dx$.

Solution:

Example 3.

Compute $\int \ln x \, dx$.

Solution:

Example 4.

Compute $\int x^2 \ln x \, dx$.

Solution:

Example 5.

Compute $\int e^x \sin x \, dx$.

Solution:

Powers of $\sin$ and $\cos$

Strategy:

- If the power of $\sin x$ is odd, save one $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$, then $u = \cos x$.
- If the power of $\cos x$ is odd, save one $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$, then $u = \sin x$.
- If both are even, use half-angle identities.

Example 1.
Compute $\int \sin^5 x \cos^2 x \, dx$.

Solution:

Example 2.

Compute $\int_0^{\pi} \sin^2 x \, dx$.

Solution:

Powers of $\tan$ and $\sec$

Recall: $\tan^2 x + 1 = \sec^2 x$.
Strategy for $\int \tan^n x \sec^m x \, dx$:

- If m is even, save $\sec^2 x$ and let $u = \tan x$.
- If n is odd, save $\sec x \tan x$ and let $u = \sec x$.

Example 3.
Compute $\displaystyle\int \tan^5 x \sec^4 x \, dx$.

Solution:

Example 4.

Compute $\int \sec^3 x \tan x \, dx$.

Solution:

Useful special integrals

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C, \qquad \int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products $\sin(ax)\cos(bx)$, etc.

Use product-to-sum identities:

$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B)+\sin(A+B)\right).$$

Example 5.
Compute $\int \sin(7x)\cos(3x)\,dx$.

Solution:

Example 6.

Compute $\int \tan^2 x \cos^3 x \, dx$.

Solution:

Standard forms

$$\sqrt{a^2 - x^2} \quad \Rightarrow \quad x = a\sin\theta, \ \sqrt{a^2 - x^2} = a\cos\theta$$
$$\sqrt{a^2 + x^2} \quad \Rightarrow \quad x = a\tan\theta, \ \sqrt{a^2 + x^2} = a\sec\theta$$
$$\sqrt{x^2 - a^2} \quad \Rightarrow \quad x = a\sec\theta, \ \sqrt{x^2 - a^2} = a\tan\theta$$

Example 1.

Compute $\int \frac{dx}{\sqrt{9 - x^2}}$.

Solution:

Example 2.

Compute $\int \sqrt{9-x^2}\,dx$.

Solution:

Example 3.
Compute $\int \frac{dx}{\sqrt{x^2+16}}$.

Solution:

Example 4.
Compute $\int \sqrt{x^2 - 25}\, dx$.

Solution:

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SECTION 7.4: PARTIAL FRACTIONS

Overview

We use partial fractions to integrate rational functions:

$$\int \frac{P(x)}{Q(x)}\,dx$$

Step 1: If $\deg P \geq \deg Q$, do long division first.
Step 2: Factor $Q(x)$ over $\mathbb{R}$.
Step 3: Decompose into partial fractions.
Step 4: Integrate term-by-term.

Decomposition templates

Distinct linear factors: $\dfrac{A}{x-a} + \dfrac{B}{x-b} + \cdots$

Repeated linear factor: $\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \cdots + \dfrac{K}{(x-a)^n}$

Irreducible quadratic: $\dfrac{Ax+B}{x^2+px+q}$

Repeated irreducible quadratic: $\dfrac{A_1x+B_1}{x^2+px+q} + \dfrac{A_2x+B_2}{(x^2+px+q)^2} + \cdots$

Example 1.

Compute $\displaystyle\int \frac{3x+5}{(x-1)(x+2)}\,dx$.

Solution:

Example 2.
Compute $\int \frac{1}{x(x-3)}\,dx$.

Solution:

Example 3.

Compute $\int \frac{5}{(x-1)^2}\,dx$.

Solution:

Example 4.

Compute $\int \frac{2x+1}{x^2+4x+8}\,dx$.

Solution:

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SECTION 7.5: STRATEGIES FOR INTEGRATION

A decision checklist

When you see $\int f(x)\,dx$, ask:

1. Can I simplify algebraically?
2. Is there an obvious substitution $u = g(x)$?
3. Is it a trig integral (powers of trig), or trig substitution?
4. Is it a rational function (partial fractions)?
5. Is it a product that suggests IBP?
6. Can I use an identity (trig, logarithms, inverse trig)?

Example 1.
Compute $\int \frac{x^2}{x^2+1}\,dx$.

Solution:

Example 2.
Compute $\int x\sqrt{x^2+9}\,dx$.

Solution:

Example 3.

Compute $\int \frac{dx}{x^2\sqrt{x^2-1}}$.

Solution:

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SECTION 7.7: APPROXIMATE INTEGRATION

Riemann sums

Partition $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$. A right Riemann sum:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x, \qquad x_i = a + i\Delta x$$

Trapezoidal Rule

$$T_n = \frac{\Delta x}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big)$$

Simpson's Rule (n even)

$$S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big)$$

Example 1.

Approximate $\int_0^2 (1 + x^3)\,dx$ using $n = 4$ trapezoids.

Solution:

Example 2.

Approximate $\int_0^2 (1 + x^3)\,dx$ using Simpson's Rule with $n = 4$.

Solution:

Error bounds (if needed)

If $|f''(x)| \le K$ on $[a, b]$, then trapezoid error satisfies

$$|E_T| \le \frac{K(b-a)^3}{12n^2}.$$

If $|f^{(4)}(x)| \le K$ on $[a, b]$, Simpson error satisfies

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$

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SECTION 7.8: IMPROPER INTEGRALS

Type 1: Infinite interval

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx$$

If the limit exists (finite), the integral converges. Otherwise it diverges.

Type 2: Infinite discontinuity

If f has a vertical asymptote at $x = a$ on $[a, b]$:

$$\int_a^b f(x)\,dx := \lim_{t\to a^+}\int_t^b f(x)\,dx$$

Example 1.

Determine whether $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution:

Example 2.

Evaluate $\int_1^{\infty} \frac{1}{x^2}\, dx$.

Solution:

Example 3.

Determine whether $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges.

Solution:

Comparison tests (very common)

If $0 \leq f(x) \leq g(x)$ and $\int g$ converges, then $\int f$ converges. If $0 \leq g(x) \leq f(x)$ and $\int g$ diverges, then $\int f$ diverges.

Example 4.

Determine whether $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx$ converges.

Solution:

Chapter 8&9

Differential Equations

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SECTION 8.1: ARC LENGTH

Last Time: Improper Integrals

- Type I: Infinite interval
- Type II: Discontinuous integrand

Arc Length Motivation

In Calc I, you saw applications of integrals: areas, volumes, work, etc.
Today we see one more: **arc length**.

Recall: the distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Deriving the Arc Length Formula

Suppose we have a curve $y = f(x)$ on an interval $[a, b]$, where f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$.

Partition $[a, b]$ and approximate the curve by line segments between points P_{i-1} and P_i. Then the length is approximated by

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|.$$

By the Pythagorean Theorem,

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

By the Mean Value Theorem, there exists $x_i^* \in [x_{i-1}, x_i]$ such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

So

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x_i^*)\right)^2}\,\Delta x_i = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx.$$

Arc Length Formula. If $f'(x)$ is continuous on $[a, b]$, then

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx.$$

Arc Length Function

The **arc length function** from $x = a$ to x is

$$s(x) = \int_a^x \sqrt{1 + \left(f'(t)\right)^2}\, dt.$$

(In particular, $s(b) = L$.)

Example 1. Find the length of the curve $y = 1 + 2x^{3/2}$ for $0 \leq x \leq 1$.

Solution:

Example 2. Find the arc length function for $y = \frac{e^x + e^{-x}}{2}$ with $P_0 = (0, 1)$. Then compute $s(2)$.

Solution:

Arc Length When $x = g(y)$

For a curve $x = g(y)$ where $g'(y)$ is continuous on $c \leq y \leq d$, the arc length is

$$L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

Example 3. Show that the circumference of a circle with radius R is $2\pi R$.

Solution:

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SECTION 9.2: DIRECTION FIELDS AND EULER'S METHOD

Warm-up: Differential Equations

A **differential equation** is an equation relating an unknown function to one or more of its derivatives.
The **order** of a differential equation is the order of the highest derivative that occurs.

Examples (all order 1):

$$y' = 2x, \qquad y' = ky, \qquad y' = xy^2.$$

Example (order 2):

$$y'' + xy' = x^3 y.$$

A **solution** is a function $y = f(x)$ that satisfies the equation.

General Solutions

A **general solution** is a solution involving constants that describes a family of functions. For example:

$$y' = 2x \Rightarrow y = x^2 + C.$$

Direction Fields

Consider a first-order differential equation of the form

$$y' = F(x, y).$$

The **direction field** (slope field) is a collection of short line segments with slope $F(x, y)$ drawn at points (x, y).
The direction field gives the *shape* of solutions; different initial values give different solutions.

Example 1. Sketch the solution of $y' = x + y$ with initial value $y(0) = 1$ using a direction field.

Solution:

Euler's Method

Idea: use the direction field and linear segments to numerically approximate the value of the solution near the initial value.

Suppose $y' = F(x, y)$, $y(x_0) = y_0$. Choose a step size h. Define

$$x_n = x_{n-1} + h, \qquad y_n \approx y(x_n).$$

Using slope $F(x_{n-1}, y_{n-1})$,

$$\Delta y \approx hF(x_{n-1}, y_{n-1}), \quad \text{so} \quad y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}).$$

Euler's Method. Approximate the solution of $y' = F(x, y)$, $y(x_0) = y_0$, with step size h by

$$x_n = x_{n-1} + h, \qquad y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}), \qquad n = 1, 2, 3, \ldots$$

Example 2. Approximate $y(2)$ for $y' = x + y$, $y(0) = 1$, using Euler's method with $h = 1$.

Solution:

Example 3. Approximate $y(2)$ for $y' = x + y$, $y(0) = 1$, using Euler's method with $h = \frac{1}{2}$.

Solution:

Example 4. Estimate the solution to $y' = x^2 + y$, $y(1) = 0$, at $x = 1.3$ with step size $h = 0.1$.

Solution:

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SECTION 9.3: SEPARABLE DIFFERENTIAL EQUATIONS

Last Time

Approximating solutions to first-order differential equations

$$y' = F(x, y), \qquad y(x_0) = y_0,$$

with direction fields and Euler's method:

$$x_n = x_{n-1} + h, \qquad y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}).$$

Separable Equations

Definition 1. *A* ***separable equation*** *is a first-order differential equation that can be written in the form*

$$\frac{dy}{dx} = F(y)g(x).$$

How to Solve

To solve $\frac{dy}{dx} = F(y)g(x)$:

1. Rewrite in differential form:
$$\frac{1}{F(y)}\,dy = g(x)\,dx.$$
2. Integrate both sides:
$$\int \frac{1}{F(y)}\,dy = \int g(x)\,dx.$$
3. Solve for y (if possible). If not, the result is an implicit solution.

Example 1. Solve $y' = -2xy^2$.

Solution:

Example 2. Solve the initial value problem $y' = xy^2,\ y(1) = -2$.

Solution:

Example 3. Solve $y' - 2xy = 6x$.

Solution:

Example 4. Solve $y' = \dfrac{6x^2}{2y + \cos(y)}$.

Solution:

Example 5. Can the following be solved by separation of variables?
(a) $y' = 3y - x^2 y$ (b) $y' = 4x + 5y + 4$ (c) $y' = e^{y-x}$

Solution:

Application: Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each member of the family orthogonally (at right angles).

Example 6. Consider the family of parabolas defined by $x = ky^2$. Find the orthogonal trajectories.

Solution:

Mixing Problems (Setup)

Want: amount of substance in a tank at time t, call it $y(t)$.
General model:

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}).$$

Example 7. A tank initially contains 100 L of water. An alcohol solution with concentration 5% flows in at 10 L/min, and the solution leaves at the same rate. How much alcohol is in the tank after t minutes?

Solution:

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SECTION 9.5: LINEAR DIFFERENTIAL EQUATIONS

Linear Equations

A **linear first-order differential equation** has the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where P and Q are continuous on an interval.

Example 1. Solve $y' + \frac{1}{x}y = 2$.

Solution:

Integrating Factor Method

We want $I(x)$ so that multiplying the equation by $I(x)$ turns the left side into a product derivative:

$$I(x)\left(y' + P(x)y\right) = (I(x)y)'.$$

Expanding:

$$Iy' + PIy = I'y + Iy' \quad \Rightarrow \quad I' = PI.$$

Thus choose

$$I(x) = e^{\int P(x)\,dx}.$$

Procedure

To solve $y' + P(x)y = Q(x)$:

1. Identify $P(x)$ and $Q(x)$.
2. Compute $I(x) = e^{\int P(x)\,dx}$.
3. Multiply: $I(x)y' + I(x)P(x)y = I(x)Q(x)$, so $(I(x)y)' = I(x)Q(x)$.
4. Integrate: $I(x)y = \int I(x)Q(x)\,dx + C$.
5. Solve for y (and use an initial condition if given).

Example 2. Solve $x\frac{dy}{dx} = x^2 + 3y$.

Solution:

Example 3. Solve the IVP $\frac{dy}{dx} + 3x^2 y = 6x^2, \quad y(0) = 1.$

Solution:

Mixing Setup Example

Example 4. A tank initially contains 100 L of water. An alcohol solution with concentration 5% flows in at 4 L/min, and the solution leaves at the 10 L/min. How much alcohol is in the tank after t minutes?

Solution:

Chapter 11
Series

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SECTION 11.1: SEQUENCES

What is a Sequence?

A sequence is a function whose domain is the natural numbers. We usually write a sequence as $\{a_n\}_{n=1}^{\infty}$ or $a_1, a_2, a_3, \ldots$.

Definition 1. *A sequence $\{a_n\}$ converges to L if the terms get arbitrarily close to L as $n \to \infty$. We write*

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L.$$

If no such L exists, the sequence diverges.

Limit Laws for Sequences

If $\lim a_n = A$ and $\lim b_n = B$, then (when defined):

$$\lim(a_n \pm b_n) = A \pm B, \qquad \lim(a_n b_n) = AB, \qquad \lim \frac{a_n}{b_n} = \frac{A}{B} \ (B \neq 0),$$

and for any constant c, $\lim(ca_n) = cA$.

Useful Facts

- If $\lim_{n\to\infty} a_n = L$, then a_n is bounded (eventually).
- If $a_n \to L$, then $|a_n| \to |L|$.
- If $a_n \to L$, then every subsequence also converges to L.

Squeeze Theorem for Sequences

Theorem 2 (Squeeze Theorem)**.** *If $a_n \le b_n \le c_n$ for all sufficiently large n, and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

Monotone Convergence Theorem

Theorem 3 (Monotone Convergence Theorem)**.** *If $\{a_n\}$ is monotone increasing and bounded above, then $\{a_n\}$ converges. If $\{a_n\}$ is monotone decreasing and bounded below, then $\{a_n\}$ converges.*

Example 1. Compute $\lim_{n\to\infty} \frac{3n^2 - 5n + 1}{2n^2 + 7}$.

Solution:

Example 2. Determine whether $a_n = (-1)^n$ converges.

Solution:

Example 3. Compute $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$.

Solution:

Example 4. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$. Show $\{a_n\}$ converges and find its limit.

Solution:

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SECTION 11.2: SERIES

Series

Given a sequence $\{a_n\}$, we can form the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Partial Sums

Define the n-th partial sum:

$$s_n = \sum_{k=1}^{n} a_k.$$

Then the series converges precisely when the sequence $\{s_n\}$ converges.

Definition 1. *We say $\sum_{n=1}^{\infty} a_n$ converges to S if*

$$\lim_{n\to\infty} s_n = S.$$

If the limit does not exist (or is infinite), the series diverges.

Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

Partial sum:

$$s_n = \sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r} \quad (r \neq 1).$$

So:

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, & |r| < 1, \\ \text{diverges}, & |r| \geq 1. \end{cases}$$

Telescoping Series

A series is telescoping if cancellation occurs in the partial sums. The strategy is:

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and then compute } \lim_{n\to\infty} s_n.$$

Divergence Test (Nth-Term Test)

Theorem 1 (Divergence Test). *If* $\lim_{n\to\infty} a_n \neq 0$ *(or the limit does not exist), then* $\sum a_n$ *diverges.*

Note: If $\lim a_n = 0$, the test is inconclusive.

Example 1. Determine whether $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and find its sum.

Solution:

Example 2. Determine whether $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ converges.

Solution:

Example 3. Evaluate $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution:

Example 4. Use the divergence test to show $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Solution:

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SECTION 11.4: COMPARISON TESTS

Setup: Positive-Term Series

The comparison tests apply to series with nonnegative terms:

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0.$$

Direct Comparison Test

Theorem 1 (Comparison Test)**.** *Suppose* $0 \leq a_n \leq b_n$ *for all* n *sufficiently large.*

- *If* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*

Limit Comparison Test

Theorem 2 (Limit Comparison Test)**.** *Suppose* $a_n \geq 0$, $b_n \geq 0$ *and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some constant c *with* $0 < c < \infty$. *Then* $\sum a_n$ *and* $\sum b_n$ *either both converge or both diverge.*

Choosing b_n

Typical choices:

p-series: $\sum \frac{1}{n^p}$, geometric: $\sum r^n$, (sometimes) known convergent/divergent benchmarks.

Example 1. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Solution:

Example 2. Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^3+5}$ converges.

Solution:

Example 3. Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ converges.

Solution:

Example 4. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

Solution:

Reminder

Comparison tests require $a_n, b_n \geq 0$. If your series has signs, first consider absolute values or a different test.

Alternating Series

An alternating series is a series whose terms alternate in sign, typically of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^n b_n,$$

where $b_n \geq 0$.

Alternating Series Test (AST)

Theorem 1 (Alternating Series Test)**.** *Suppose* $b_n \geq 0$. *If*

$$\text{(i) } b_{n+1} \leq b_n \text{ for all sufficiently large } n \quad \text{and} \quad \text{(ii) } \lim_{n\to\infty} b_n = 0,$$

then the alternating series $\sum(-1)^{n-1}b_n$ *converges.*

If either condition fails, the test is inconclusive (or the series diverges by the divergence test if the terms do not go to 0*).*

Example 1. Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges.

Solution:

Example 2. Determine whether $\sum_{n=3}^{\infty}(-1)^n \frac{\ln n}{n}$ converges.

Solution:

Example 3. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$ converges.

Solution:

Alternating Series Estimation Theorem (Error Bound)

If $\sum(-1)^{n-1}b_n$ satisfies AST and S is its sum, then the remainder after n terms,

$$R_n = S - s_n,$$

satisfies

$$|R_n| \leq b_{n+1}.$$

Example 4. How many terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ are needed to approximate the sum within 0.01?

Solution:

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SECTION 11.6: ABSOLUTE CONVERGENCE (RATIO / ROOT TESTS)

Absolute Convergence

Given any series $\sum a_n$, consider the corresponding series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Definition 1. *A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.*

Definition 2. *A series $\sum a_n$ is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.*

Theorem 3. *If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*

Proof idea (as in notes). Since $|a_n| \geq 0$,

$$0 \leq a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n|.$$

If $\sum |a_n|$ converges, then $\sum 2|a_n|$ converges, and by comparison $\sum (a_n + |a_n|)$ converges. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series, so it converges.

Example 1. Does $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ converge?

Solution:

The Ratio Test

Theorem 4 (Ratio Test)**.** *Given a series* $\sum a_n$*, let*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(when the limit exists).

- *If* $L < 1$*, then* $\sum a_n$ *converges* *<u>absolutely</u>.*
- *If* $L = 1$*, the Ratio Test is inconclusive.*
- *If* $L > 1$ *(or* $L = \infty$*), then* $\sum a_n$ *diverges.*

Example 2. Use the Ratio Test to determine whether $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$ converges.

Solution:

The Root Test

Theorem 5 (Root Test)**.** *Given a series $\sum a_n$, let*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

- *If $L < 1$, then $\sum a_n$ converges absolutely.*
- *If $L = 1$, the Root Test is inconclusive.*
- *If $L > 1$ (or $L = \infty$), then $\sum a_n$ diverges.*

Example 3. Use the Root Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ converges.

Solution:

Example 4. Determine whether $\sum_{n=1}^{\infty}\left(\frac{2n}{n+1}\right)^{5n}$ converges.

Solution:

Fun facts (rearrangements)

- If $\sum a_n$ is absolutely convergent and $\sum a_n = S$, then *any rearrangement* of the terms still sums to S.
- If $\sum a_n$ is conditionally convergent, then for any real number r, we can rearrange the terms so the rearranged series sums to r.

In particular (preview for later),

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad (|x|<1),$$

and at $x=1$,

$$\ln 2 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

A certain rearrangement groups terms to produce $\frac{1}{2}\ln 2$.

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SECTION 11.7: STRATEGIES FOR SERIES

Series Tests (Quick List)

1. **Geometric:** $\sum_{n=1}^{\infty} ar^{n-1}$.

$$|r| < 1 \Rightarrow \text{converges to } \frac{a}{1-r}, \qquad |r| \geq 1 \Rightarrow \text{diverges.}$$

2. **Telescoping:** middle terms cancel. Compute partial sums $s_k = \sum_{n=1}^{k} a_n$ and take $\lim_{k\to\infty} s_k$ (if it exists).

3. p**-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

$$p > 1 \Rightarrow \text{converges}, \qquad p \leq 1 \Rightarrow \text{diverges.}$$

   (Harmonic series is $p = 1$: $\sum \frac{1}{n}$ diverges.)

4. **Divergence Test:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. (If $\lim a_n = 0$, the test is inconclusive.)

5. **Comparison Test (positive series):** If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

6. **Limit Comparison Test (positive series):** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ have the same behavior.

7. **Alternating Series Test (AST):** If $b_n \geq 0$, $b_{n+1} \leq b_n$ eventually, and $\lim b_n = 0$, then $\sum (-1)^n b_n$ converges.

8. **Absolute Convergence:** If $\sum |a_n|$ converges, then $\sum a_n$ converges *absolutely*.

$\sum \|a_n\|$	$\sum a_n$	Conclusion
conv	conv	conv absolutely
div	conv	conv conditionally
div	div	diverges

9. **Ratio Test:** $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$ absolutely convergent; if $L = 1$ inconclusive; if $L > 1$ diverges.

10. **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. If $L < 1$ absolutely convergent; if $L = 1$ inconclusive; if $L > 1$ diverges.

A Practical Strategy (Flowchart in words)

Given $\sum a_n$:

1. **Is it a series you recognize?** Try: geometric, telescoping, p-series, alternating series.
2. **Always consider the Divergence Test first.** If $\lim a_n \neq 0$, stop: it diverges.
3. **If** $\lim a_n = 0$**, ask: "What does it look like?"**
   - Looks like a known comparison target (e.g., $1/n^p$, geometric-like): try Comparison or Limit Comparison.
   - Has factorials: try Ratio Test.
   - Has powers like $(\cdot)^n$ or something natural for an n-th root: try Root Test.
4. **If the question asks absolute vs conditional convergence:**
   - Option A: do Ratio or Root Test (quick when applicable).
   - Option B: test $\sum |a_n|$ using any convergence test, then decide absolute/conditional.

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SECTION 11.7B: STRATEGIES FOR SERIES (CONTINUED)

How to Choose a Test

When given a series $\sum a_n$, your goal is to determine whether it **converges absolutely**, **converges conditionally**, or **diverges**.

- **Step 0: Recognition.** Check quickly for geometric series, telescoping series, or p-series.
- **Step 1: Divergence Test.** If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.
- **Step 2: Positive-term series.** If all terms are positive, consider the Comparison Test or Limit Comparison Test.
- **Step 3: Alternating series.** If the series alternates, test absolute convergence first. If that fails, try the Alternating Series Test.
- **Step 4: Factorials or exponentials.** If you see $n!$, a^n, or powers raised to n, try the Ratio Test or Root Test.

Directions. For each series below, determine whether it converges absolutely, converges conditionally, or diverges. Clearly justify your answer by naming the test used.

Problems

Problem 1.

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$$

Problem 2.

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$$

Problem 3.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$$

Problem 4.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$$

Problem 5.

$$\sum_{n=1}^{\infty} \frac{e^n}{n^2}$$

Problem 6.

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$$

Problem 7.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Problem 8.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$$

Problem 9.

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n)!}$$

Problem 10.

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

Problem 11.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n} \right)$$

Problem 12.

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$$

Problem 13.

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

Problem 14.

$$\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$$

Problem 15.

$$\sum_{k=1}^{\infty} \frac{2^k \cdot 3^{k+1}}{k^k}$$

Problem 16.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$$

Problem 17.

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

Problem 18.

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}-1}$$

Problem 19.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

Problem 20.

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$$

Problem 21.

$$\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$$

Problem 22.

$$\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$$

Problem 23.

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

Problem 24.

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$$

Problem 25.

$$\sum_{n=1}^{\infty} \frac{4 - \cos n}{\sqrt{n}}$$

Problem 26.

$$\sum_{n=1}^{\infty} \frac{8 + (-1)^n n}{n}$$

Problem 27.

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

Problem 28.

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$$

Problem 29.

$$\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$$

Problem 30.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

Problem 31.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$$

Problem 32.

$$\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$$

Problem 33.

$$\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$$

Problem 34.

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^n}$$

Problem 35.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^n}$$

Problem 36.

$$\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$$

Problem 37.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

Problem 38.

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{1/3}$$

Problem 39.

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$$

Problem 40.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+1) 3^n}{2^{2n+1}}$$

Problem 41.

$$\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{\ln n}$$

Extra Work

Extra Work

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SECTION 11.8: POWER SERIES

Power Series

Definition 1. *A power series centered at a is an infinite series of the form*

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

where c_n are constants.

Fact 2. *Substituting $x = a$ gives $(x-a)^n = 0$ for $n \geq 1$. Thus every power series converges at $x = a$, and its value is c_0.*

Theorem 3 (Radius and Interval of Convergence)**.** *For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there exists a number R (possibly 0 or ∞) such that:*

- *The series converges absolutely for $|x-a| < R$,*
- *The series diverges for $|x-a| > R$,*
- *At the endpoints $|x-a| = R$, convergence must be tested separately.*

R is the radius of convergence. The set of x-values for which the series converges is the interval of convergence.

Finding R (usually Ratio Test)

If $c_n \neq 0$ eventually, apply the Ratio Test to

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

Often you compute

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = |x-a| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|.$$

Then require $L < 1$ to get $|x-a| < R$.

Example 1. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{n}{2^n}(x+2)^n$.

Solution:

Example 2. Find the radius of convergence of $\sum_{n=0}^{\infty} n!\,x^n$.

Solution:

Key idea

A power series behaves like a polynomial inside its interval of convergence (we can add/subtract, integrate, differentiate term-by-term—next section formalizes this).

A geometric-series backbone

Fact 1 (Geometric Series)**.** *For* $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

A common strategy:

1. Start with $\sum r^n = \frac{1}{1-r}$,
2. Substitute to match a target expression,
3. Differentiate or integrate to generate new series,
4. Track the interval of convergence.

Example 1. Write $\frac{1}{1+x^2}$ as a power series centered at 0, and find its interval of convergence.

Solution:

Example 2. Use a power series to find a series for $\arctan(x)$ and state its interval of convergence.

Solution:

Example 3. Find a power series for $\ln(1 + x)$ centered at 0.

Solution:

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SECTION 11.10: TAYLOR SERIES

Taylor Polynomials

Definition 1. *The nth Taylor polynomial of f about $x = a$ is*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

When $a = 0$, it is called the Maclaurin polynomial.

Taylor Series

Definition 2. *The Taylor series of f about a is*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

provided the series converges. If it converges to $f(x)$, then f is represented by its Taylor series (on that interval).

Fact 3 (Common Maclaurin series)**.**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\textit{all } x)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (\textit{all } x)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (\textit{all } x)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (|x| < 1)$$

Example 1. Find the Maclaurin series for $\cos x$, and write the 4th-degree Maclaurin polynomial $P_4(x)$.

Solution:

Example 2. Find the Taylor series of $f(x) = e^x$ about $a = 2$.

Solution:

Remainder idea (for error bounds)

If $f(x) = P_n(x) + R_n(x)$, then $R_n(x)$ measures the error from truncation.

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SECTION 11.11: APPLICATIONS OF TAYLOR POLYNOMIALS

Linearization vs. higher-order approximations

Taylor polynomials approximate $f(x)$ near $x = a$:

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Error bounds (Lagrange form of the remainder)

Theorem 1 (Taylor's Theorem with Remainder)**.** *Suppose f has $n+1$ derivatives on an interval containing a and x. Then*

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Fact 2 (Practical bound)**.** *If $|f^{(n+1)}(t)| \le M$ for all t between a and x, then*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Example 1. Approximate $\sqrt{1.02}$ using a Taylor polynomial, and estimate the error.

Solution:

Example 2. Use the Maclaurin series for $\sin x$ to approximate $\sin(0.1)$ and bound the error using the next term.

Solution:

Example 3. Sum the series

$$\sum_{n=0}^{\infty} \frac{5^n}{n!} \qquad\qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Solution:

Example 4. Compute

$$\lim_{x \to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

Solution:

Chapter 10
Other Coordinates

Motivation

Sometimes it is easier to describe a curve by giving both coordinates as functions of a parameter t:

$$x = f(t), \qquad y = g(t).$$

The pair $(x(t), y(t))$ traces out a curve as t varies.

Parameter Interval and Orientation

The same geometric curve can be traced with different parametrizations, different intervals, or different directions (orientation).

Example 1. Eliminate the parameter to identify the curve: $x = t^2 + 1,\ y = 2t$.

Solution:

Example 2. Find a parametrization of the circle $x^2 + y^2 = 9$ that traces it once counterclockwise.

Solution:

Parametric Graphing Notes

- Plot key points by evaluating t.
- Determine direction by increasing t.
- Look for symmetry and periodic behavior.

Example 3. Sketch the curve $x = \cos t,\ y = \sin(2t)$ for $0 \le t \le 2\pi$.

Solution:

First Derivative

If $x = f(t)$ and $y = g(t)$, then by the Chain Rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{(provided } \frac{dx}{dt} \neq 0\text{).}$$

Example 1. Given $x = t^2 + 1$ and $y = t^3 - 3t$, find $\frac{dy}{dx}$.

Solution:

Second Derivative

Differentiate $\frac{dy}{dx}$ with respect to t, then divide by $\frac{dx}{dt}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Example 2. For $x = t^2 + 1$ and $y = t^3 - 3t$, find $\frac{d^2y}{dx^2}$.

Solution:

Tangent Lines

A horizontal tangent occurs when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.
Example 3. Find values of t where the curve $x = t^2 - 4t,\ y = t^2 - 1$ has horizontal tangents.

Solution:

Area Under a Parametric Curve

If $x = f(t)$ is increasing on $[a, b]$, then the area under the curve is

$$A = \int y\,dx = \int_a^b y(t)\,x'(t)\,dt.$$

Example 4. Find the area under one arch of the cycloid $x = t - \sin t,\; y = 1 - \cos t$, $0 \le t \le 2\pi$.

Solution:

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SECTION 10.3: POLAR COORDINATES

Coordinate Systems

In polar coordinates, a point is described by (r, θ), where

$$r = \text{distance from the origin}, \qquad \theta = \text{angle from the positive } x\text{-axis}.$$

Conversion:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}.$$

Non-uniqueness

(r, θ) is not unique:

$$(r, \theta) \equiv (r, \theta + 2\pi k), \qquad (r, \theta) \equiv (-r, \theta + \pi).$$

Example 1. Convert $(r, \theta) = (2, \frac{\pi}{3})$ to rectangular coordinates.

Solution:

Example 2. Convert $x^2 + y^2 = 4x$ to a polar equation and identify the curve.

Solution:

Basic Polar Curves

- $r = a$: circle of radius a centered at origin.
- $\theta = \theta_0$: line through origin with angle θ_0.
- $r = a\cos\theta$ or $r = a\sin\theta$: circle through origin.

Example 3. Sketch $r = 2\sin\theta$.

Solution:

Area in Polar Coordinates

If a curve is given by $r = f(\theta)$ for $a \le \theta \le b$, then the area of the region it bounds is

$$A = \frac{1}{2}\int_a^b \left(f(\theta)\right)^2 \, d\theta.$$

Example 1. Find the area inside $r = 2\cos\theta$.

Solution:

Arc Length in Polar Coordinates

If $r = f(\theta)$ with continuous derivative, the arc length from a to b is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 2. Find the length of the curve $r = 3$ for $0 \le \theta \le 2\pi$.

Solution:

Slope of a Polar Curve

If $x = r\cos\theta,\ y = r\sin\theta$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta},$$

when the denominator is nonzero.

Example 3. Find $\frac{dy}{dx}$ for $r = 2\cos\theta$.

Solution: