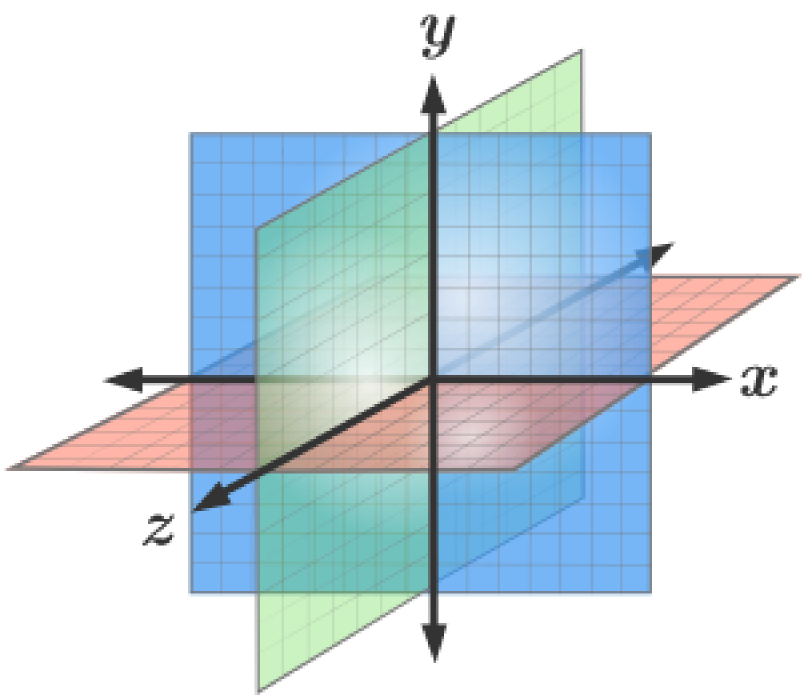
University of Notre Dame Calculus III

3-dimensional Coordinate System

The 3-dimensional coordinate system we use are coordinates on $\mathbb{R}^3$. The coordinate is presented as a triple of numbers: (a,b,c). In the Cartesian coordinate system we have an origin $(0,0,0)$, and three axis: the x-, y-, z-axes. These 3 axes are perpendicular to each other and their positive directions satisfy the "right hand rule": point your index finger on your right hand along the x-axis, curl it toward the y-axis, then your "thumb up" will point along the z-axis. Examples of properly drawn axes are:



To locate the point P which has coordinates (a,b,c): move a units in the x-direction, b in the y-direction, and c in the z-direction.

Example 1. Plot $(2, 1, 3)$:

Solution:

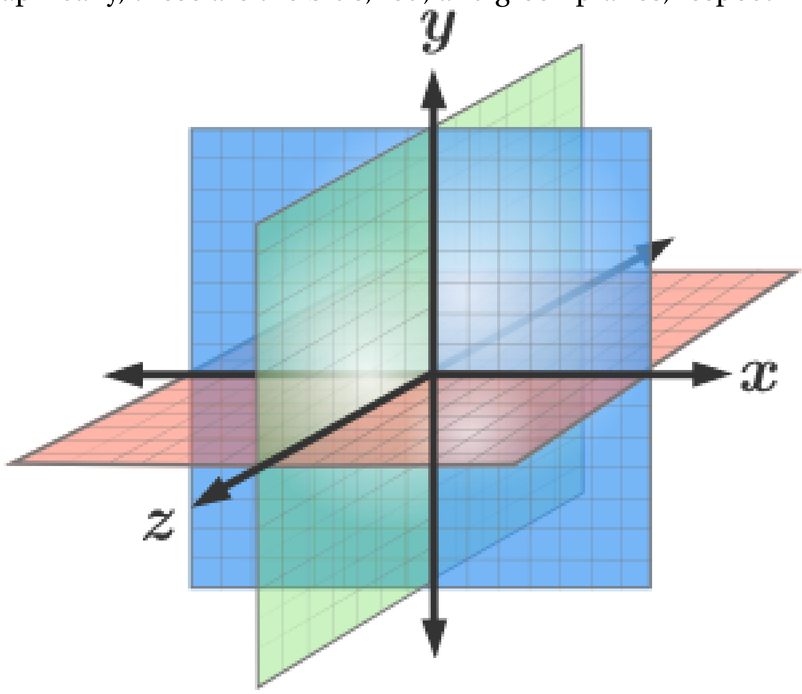
Example 2. What would the equation $z = 3$ represent in $\mathbb{R}^3$?

Solution:

Example 3. How about $y = x^2$?

Solution:

The coordinate planes are the xy-, xz-, and yz-planes, which are represented by $z = 0$, $y = 0$, and $x = 0$ respectively. Graphically, these are the blue, red, and green planes, respectively.



We can also talk about "projecting" onto the coordinate planes. This is done by setting the appropriate coordinate to 0.

The projection of (a, b, c) onto the:

- xy-plane is $(a, b, 0)$
- xz-plane is $(a, 0, c)$
- yz-plane is $(0, b, c)$

Just as in the plane, we can talk about the distance between points. Applying the Pythagorean Theorem twice, we arrive at the distance formula.

Formula 1 (Distance Formula in $\mathbb{R}^3$)**.** *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. The distance from P_1 to P_2 is*

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(Note the book uses $|P_1P_2|$ instead of $d(P_1, P_2)$.)

Consider the point (h, k, l). Suppose we want an equation for the collection of points which are distance r away from (h, k, l). Using the distance formula, we know any point (x, y, z) satisfying this criteria satisfies:

$$r = d((h, k, l), (x, y, z)) = \sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2}$$

This set of points is the sphere with radius r and center (h, k, l). Squaring both sides of the equation, we arrive at a more friendly equation for the sphere.

$$\boxed{(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2}$$

Ex: The describe the region defined by the inequalities

$$x^2 + y^2 + z^2 \leq 4 \qquad\qquad x^2 + y^2 \geq 1$$

This looks like a solid ball of radius 2, centered at the origin with a whole of radius 1 drilled through it along the z-axis.

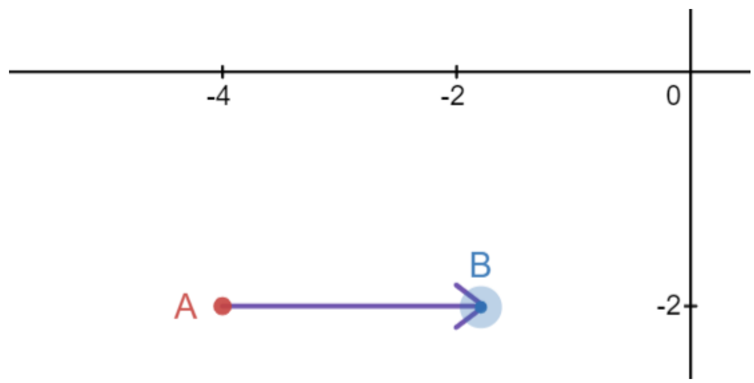
Extra Example Problem

1. Show $x^2 + y^2 + z^2 = 4x - 2y$ is a sphere.
2. Draw the region represented by $x^2 + y^2 + z^2 = 4x - 2y$ and $x \geq 2$.
3. Draw the region represented by $0 \leq z \leq 4$ and $y = x^2$.
4. What is the distance between $(1,2,3)$ and $(0,-1,2)$.

Vectors

Definition 2. *A vector is an object with direction and magnitude. There is one exception to this definition, the zero vector,* $\mathbf{0}$*, which has magnitude* 0 *has no specified direction.*

Suppose a particle moves from a point A to a point B along a straight line. Then the displacement vector, written $\mathbf{AB}$, can be visualized as an arrow from A to B, visually:



If the points have coordinates $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ we can represent $\mathbf{AB}$ as

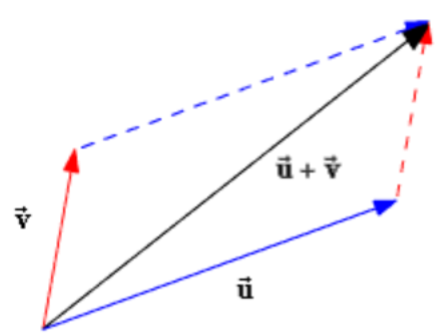
$$\mathbf{AB} = B - A = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

(this works for points in $\mathbb{R}^2$ as well)

Vector Operations

(Everything here is written for vectors in $\mathbb{R}^2$, but works in $\mathbb{R}^3$ as well)

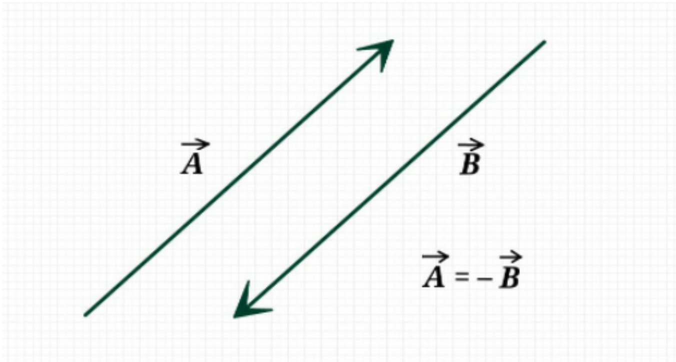
Vector Addition $\mathbf{u} + \mathbf{v}$ - Place the tail of $\mathbf{v}$ on the tip of $\mathbf{u}$ then $\mathbf{u} + \mathbf{v}$ starts at the tail of $\mathbf{u}$ and ends at the tip of $\mathbf{v}$



If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ then

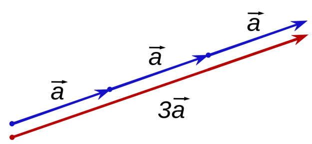
$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

Negative $\mathbf{v}$ and $-\mathbf{v}$ point in the opposite directions, but with same magnitude



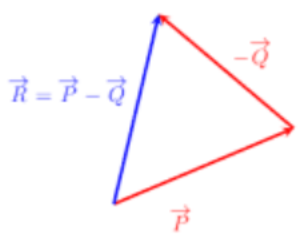
$$-\mathbf{v} = \langle -v_1, -v_2 \rangle$$

Scalar Multiplication $c\mathbf{v}$ - Scale the size of $\mathbf{v}$ by $|c|$ and point $\mathbf{v}$. If $c < 0$ then point $\mathbf{v}$ in the opposite direction. Let $c \in \mathbb{R}$ then:



$$c\mathbf{v} = \langle cv_1, cv_2 \rangle$$

Vector Subtraction $\mathbf{u} - \mathbf{v}$ - Put the vectors tail to tail then $\mathbf{u} - \mathbf{v}$ is from the head of $\mathbf{v}$ to the head of $\mathbf{u}$.



$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$$

Magnitude of a Vector

In $\mathbb{R}^3$, $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Algebraic Properties of Vectors:

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
6. $(cd)\mathbf{a}=c(d\mathbf{a})$
7. $1\mathbf{a}=\mathbf{a}$

Given any vector $\mathbf{v}=\langle a,b,c\rangle$, using the rules above, we can write

$$\mathbf{v}=\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle = a\hat{i}+b\hat{j}+c\hat{k}$$

where $\hat{i}=\langle 1,0,0\rangle$, $\hat{j}=\langle 0,1,0\rangle$, and $\hat{k}=\langle 0,0,1\rangle$ are called standard basis vectors in $\mathbb{R}^3$ (likewise, $\hat{i}=\langle 1,0\rangle$ and $\hat{j}=\langle 0,1\rangle$ are the standard basis vectors for $\mathbb{R}^2$). The coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$ are called the components of $\mathbf{v}$.

Definition 3. *A unit vector is a vector of magnitude* 1. *(I will usually denote unit vectors with a hat instead of an arrow.)*

Given a vector $\mathbf{v}\neq\mathbf{0}$, one can find the unit vector in the direction of $\mathbf{v}$ by multiplying by $\frac{1}{||\mathbf{v}||}$, i.e.

$$\hat{v}=\frac{1}{||\mathbf{v}||}\mathbf{v}$$

is a unit vector in the direction of $\mathbf{v}$. Given a vector's magnitude and direction (angle it makes with positive x-axis) we can recover the vector: If $\mathbf{v}$ is the vector, $||\mathbf{v}||$ its magnitude and direction θ, $\mathbf{v}$ can be written:

$$\mathbf{v}=||\mathbf{v}||\cos\theta\,\hat{i}+||\mathbf{v}||\sin\theta\,\hat{j}$$

Of course, this is only true for 2 dimensional vectors. The procedure is a bit different in higher dimensions.

Extra Example Problems

1. What is the unit direction of the vector $\langle -3, 0, 2 \rangle$?

2. Are the vectors $\langle -3, 0, 2 \rangle$ and $\langle 3, 0, -2 \rangle$ parallel? Why or why not? What about the vectors $\langle -3, 0, 2 \rangle$ and $\langle 3, 1, -2 \rangle$?

3. Is the vector $\langle 9\pi, 0, -6\pi \rangle$ a scalar multiple of $\langle -3, 0, 2 \rangle$. If not why? If so what is the scalar? What about the vectors $\langle -3, 0, 2 \rangle$ and $\langle 3, 1, -2 \rangle$?