University of Notre Dame Calculus III

LECTURE 2: DOT AND CROSS PRODUCTS

Dot Product

We've discussed how to add, subtract, and multiply vectors by a scalar, but what about multiplying vectors? Should it produce a number, or a vector? This first product will produce a scalar:

Definition 1. *For* $\boldsymbol{u} = \langle u_1, u_2, u_3\rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$*, the* $\underline{\text{dot product}}$ *of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *is*

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$$

The dot product is sometimes called a $\underline{\text{scalar}}$ or $\underline{\text{inner}}$ $\underline{\text{product}}$. (The dot product for 2D vectors is defined similarly.)

Properties of the Dot Product

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be vectors and c a scalar.

1. $\mathbf{a}\cdot\mathbf{a} = ||\mathbf{a}||^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Suppose the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is θ, then another interpretation of the dot product is:

$$\boxed{\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta}$$

This can be reversed to find the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$

$$\boxed{\theta = \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}\right)}$$

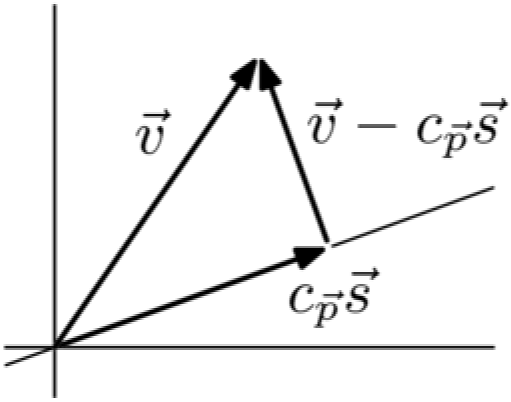
Two vectors are called $\underline{\text{perpendicular}}$ or $\underline{\text{orthogonal}}$ if their dot product is 0 (i.e. $\theta = 90^\circ$)

$$\mathbf{u}\perp\mathbf{v} \iff \mathbf{u}\cdot\mathbf{v} = 0$$

Projections

Let's say we have two vectors **u** and **v**. A question we could ask is "how much does **v** point in the direction of **u**?" or "what is the piece of **v** in the **u**-direction?"

The answer to the first question is called the scalar projection of **v** onto **u**: $\text{comp}_{\mathbf{u}}\mathbf{v}$:



Trigonometry tells us $\text{comp}_{\mathbf{u}}\mathbf{v} = ||\mathbf{v}||\cos\theta$. Recall that $\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta$, so

$$\boxed{\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||}}$$

(Notice that this number is negative if $\theta > 90°$)

The answer to the second question is the vector which is the "shadow" of **v** on **u**. It is called the vector projection of **v** onto **u**.

This vector is parallel to **u** and its length is $\text{comp}_{\mathbf{u}}\mathbf{v}$ so a formula for it is

$$\text{proj}_{\mathbf{u}}\mathbf{u} = (\text{comp}_{\mathbf{u}}\mathbf{v})\frac{\mathbf{u}}{||\mathbf{u}||} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||}\right)\frac{\mathbf{u}}{||\mathbf{u}||}$$

so

$$\boxed{\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\right)\mathbf{u}}$$

Example 1. Find the vector projection of $\mathbf{v} = \langle 0, 1, \frac{1}{2} \rangle$ onto $\langle 2, -1, 4 \rangle$.

Solution:

$$\begin{aligned}
\text{proj}_{\mathbf{u}}\mathbf{v} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\right)\mathbf{u} = \left(\frac{(2)(0)+(-1)(1)+(4)(1/2)}{(\sqrt{(2)^2+(-1)^2+(4)^2})^2}\right)\langle 2,-1,4\rangle \\
&= \left(\frac{0-1+2}{4+1+16}\right)\langle 2,-1,4\rangle \\
&= \frac{1}{21}\langle 2,-1,4\rangle
\end{aligned}$$

An Application: Work

Let's say a constant force $\mathbf{F}$ moves an object from the point P to the point Q. The displacement vector of the object is $\mathbf{d} = \mathbf{PQ}$. The amount of work $\mathbf{F}$ does in moving the object is the product of the component of $\mathbf{F}$ in the direction of $\mathbf{d}$ (i.e. $\text{comp}_{\mathbf{d}}\mathbf{F}$) and the displacement distance (i.e. $||\mathbf{d}||$). So, if θ is the angle between $\mathbf{F}$ and $\mathbf{d}$, we have

$$\boxed{\text{Work} = \text{comp}_{\mathbf{d}}\mathbf{F}||\mathbf{d}|| = (||\mathbf{F}||\cos\theta)||\mathbf{d}|| = \mathbf{F}\cdot\mathbf{d}}$$

Example 2. A child pulls a red wagon a distance of $200m$ by exerting a force of $100N$ at 20° above the horizontal. How much work has the child done in moving the wagon?

Solution:

$$\begin{aligned}
W &= (||\mathbf{F}||\cos\theta)||\mathbf{d}|| = ((100\cos 20^\circ)N)(200m) \\
&= 20000\cos 20^\circ J \\
&\approx 18.794kJ
\end{aligned}$$

Extra Example Problems

1. Compute $\langle 1,2,3\rangle \cdot \langle -1,2,-3\rangle$
2. Prove that $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$.
3. Find the $\cos(\theta)$ where θ is the angle between $\langle 1,2,3\rangle$ and $\langle -1,2,-3\rangle$.
4. Find the projection of $\langle 1,1,1\rangle$ onto $\langle 3,0,4\rangle$.

Cross Product

Suppose we are given two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. We would like to create a new vector $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ out of them so that $\mathbf{u}, \mathbf{v} \perp \mathbf{w}$. The desired conditions give us two equations:

$$\begin{cases} \mathbf{u}\cdot\mathbf{w} = 0 \\ \mathbf{v}\cdot\mathbf{w} = 0 \end{cases}$$

This actually has a whole family of solutions, one of which is

$$\mathbf{w} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$$

$\mathbf{w}$ is called the cross product of $\mathbf{u}$ and $\mathbf{v}$ and is written $\mathbf{u}\times\mathbf{v}$. We have a simpler way of computing the cross products than solving the above system or memorizing the above formula. It uses determinants. The cross product is also called the vector product.

Determinants

For a 2 × 2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3 × 3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Using the unit vector notation we can write the cross product as

$$\boxed{\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}$$

⚠**WARNING!** Whereas the dot product can be taken any two vectors of the same dimension, the cross product only makes sense in dimension 3.

Example 3. Find the cross product of $\mathbf{u} = \langle 1, 3, -2\rangle$ and $\mathbf{v} = \langle 2, 4, 6\rangle$

Solution:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 2 & 4 & 6 \end{vmatrix} = \hat{i}(18 - (-8)) - \hat{j}(6 - (-4)) + \hat{k}(4 - 6)$$
$$= \langle 26, -10, -2\rangle$$

Before, to check whether two nonzero vectors are parallel we need to find a constant c such that $\mathbf{u} = c\mathbf{v}$. The cross product gives us an easier way.

Theorem 2. *Two nonzero vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are parallel if and only if* $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{0}$

Proof. If $\mathbf{u}$ is parallel to $\mathbf{v}$, then $\mathbf{u} = c\mathbf{v}$ for some $c \in \mathbb{R}$. So $\mathbf{u} = c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$, thus

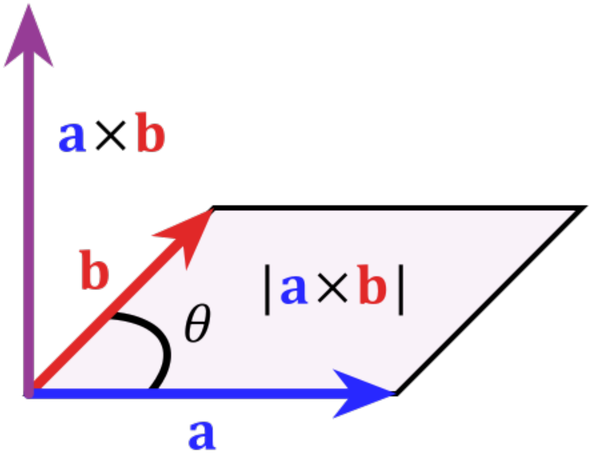
$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cv_1 & cv_2 & cv_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \hat{i}(cv_2 v_3 - cv_3 v_2) - \hat{j}(cv_1 v_3 - cv_3 v_1) + \hat{k}(cv_1 v_2 - cv_2 v_1) \\
&= \mathbf{0}
\end{aligned}$$

□

Theorem 3. *If θ is the angle between* **u** *and* **v** *(so $0 \leq \theta \leq \pi$), then*

$$\boxed{||\boldsymbol{u} \times \boldsymbol{v}|| = ||\boldsymbol{u}||\,||\boldsymbol{v}||\sin\theta}$$

This theorem actually also has a nice geometrical application: Given two victors **u** and **v**, we get the parallelogram that they span

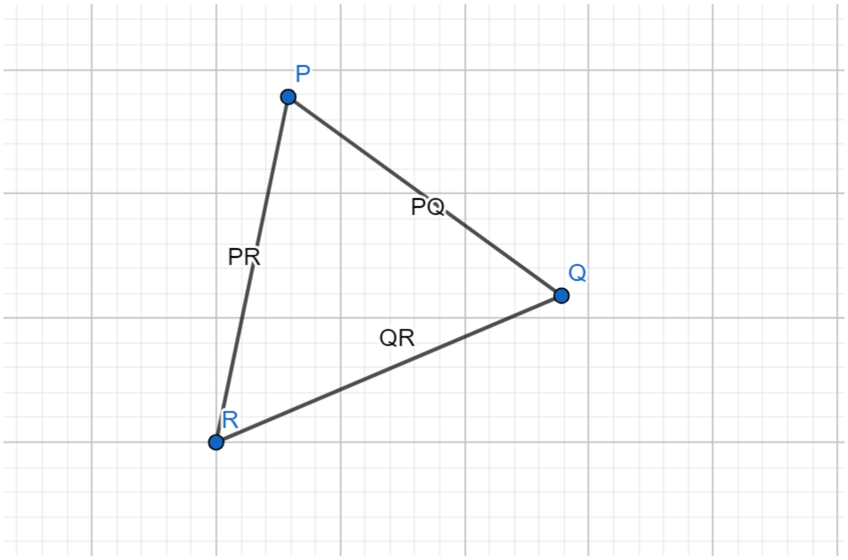


the area of which is $A = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta = ||\mathbf{u} \times \mathbf{v}||$.

Example 4. Find the area of the triangle with vertices $P = (0,0,-3)$, $Q = (4,2,0)$, and $R = (3,3,1)$

Solution:

Say the points are arranged as



Notice that the triangle $\triangle PQR$ has half the area of the parallelogram spanned by **PQ** and **PR**. So,

$$\text{Area of } \triangle PQR = \frac{1}{2}||\mathbf{PQ} \times \mathbf{PR}||$$

PQ $= Q - P =$ **4,2,3** and **PR** $=$ **3,3,4** so

$$\begin{aligned}\text{Area} &= \frac{1}{2}\left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 2 & 3\\ 3 & 3 & 4\end{vmatrix}\right\| \\ &= \frac{1}{2}||(8-9)\hat{i} - (16-9)\hat{j} + (12-6)\hat{k}|| \\ &= \frac{1}{2}||\langle -1,-7,6\rangle|| \\ &= \frac{1}{2}\sqrt{1+49+36} = \frac{1}{2}\sqrt{86}\end{aligned}$$

Using the properties of the cross product we have so far, we have the following

$$\begin{matrix}\hat{i}\times\hat{j} = \hat{k} & \hat{j}\times\hat{k} = \hat{i} & \hat{k}\times\hat{i} = \hat{j}\\ \hat{j}\times\hat{i} = -\hat{k} & \hat{k}\times\hat{j} = -\hat{i} & \hat{i}\times\hat{k} = -\hat{j}\end{matrix}$$

This can be remembered as a cyclic property. Moving in the direction of the arrows, no problem, moving against the arrows creates a minus sign in the answer.
Notice that this establishes that × is not commutative. Furthermore

$$(\hat{i}\times\hat{i})\times\hat{j} = \mathbf{0}\times\hat{j} = \mathbf{0} \quad \hat{i}\times(\hat{i}\times\hat{j}) = \hat{i}\times\hat{k} = -\hat{j}$$

meaning × is not even associative!

So, what properties are true?

Properties of the Cross Product

Let **a**, **b**, **c** be vectors and c a scalar. Then

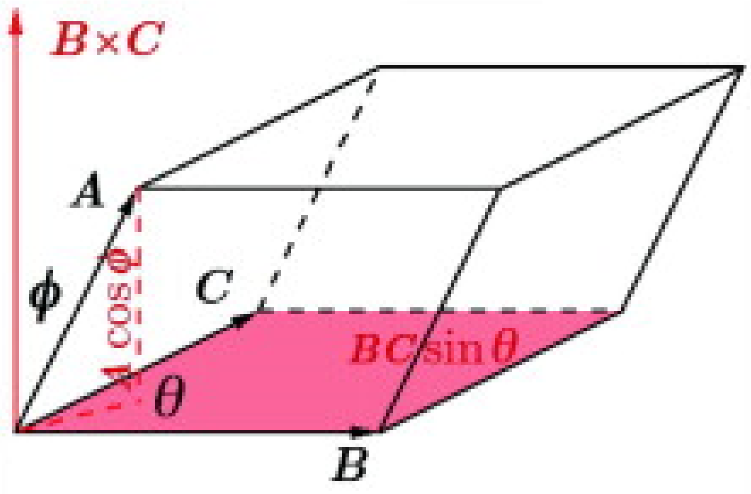
1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $(c\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(c\mathbf{b})$
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
4. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
5. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$
6. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Triple Product

Given 3 vectors **u**, **v**, **w** the triple scalar product, is the product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, a scalar, and can be computed as a determinant with the 3 vectors as rows:

$$\boxed{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

A valid question to ask is "what is the purpose of this product?" The point is the following: Just as 2 non-parallel, non-zero vectors span a parallelogram, 3 such vectors (in this they need to be pairwise non-parallel, which means non-coplaner) will span a parallelepiped:



The volume of a parallelepiped is Vol $= A \cdot h$ where A is the area of the base and h is the height. We already know $A = ||\mathbf{b} \times \mathbf{c}||$. We can find h with a little geometry; $h = ||\mathbf{a}|| \, |\cos\theta|$ (we need to use $|\cos\theta|$ in case $\theta > \frac{\pi}{2}$). This means that Vol $= A \cdot h = ||\mathbf{b} \times \mathbf{c}||(||\mathbf{a}|| \, |\cos\theta|)$. This is equivalent to

$$\text{Vol} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

The triple product has another use: checking whether 3 vectors are coplaner. Think about it geometrically. If the 3 vectors are coplanar, then the volume of the parallelepiped should be 0 since there is no "thrid direction". This gives us:

Three nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Example 5. Determine whether $\mathbf{u} = \langle 1, 5, -2 \rangle$, $\mathbf{v} = \langle 3, -1, 0 \rangle$, and $\mathbf{w} = \langle 5, 9, -4 \rangle$ are coplanar.

Solution:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} \\ &= 1(4-0) - 5(-12-0) - 2(27+5) \\ &= 4 + 60 - 64 \\ &= 0 \end{aligned}$$

So these vectors are coplanar.

An Application

Torque is created by applying a force to an object at a point given by a position vector, for example using a wrench to tighten a bolt. Torque is a measure of the tendency of the object to rotate about a pivot point (from which the position vector radiates). If the position vector is **r** and the force is **F**, the torque vector is:

$$\tau = \mathbf{r} \times \mathbf{F}$$

$$||\tau|| = ||\mathbf{r}||\;||\mathbf{F}||\sin(\theta)$$

Extra Example Problems

1. Compute $\langle 1,2,3\rangle \times \langle -1,2,-3\rangle$.
2. Find the area of the triangle PQR and a vector orthogonal to the plane containing PQR if P is $(1,0,-1)$, Q is $(2,4,5)$, and R is $(3,1,7)$.