University of Notre Dame Calculus III

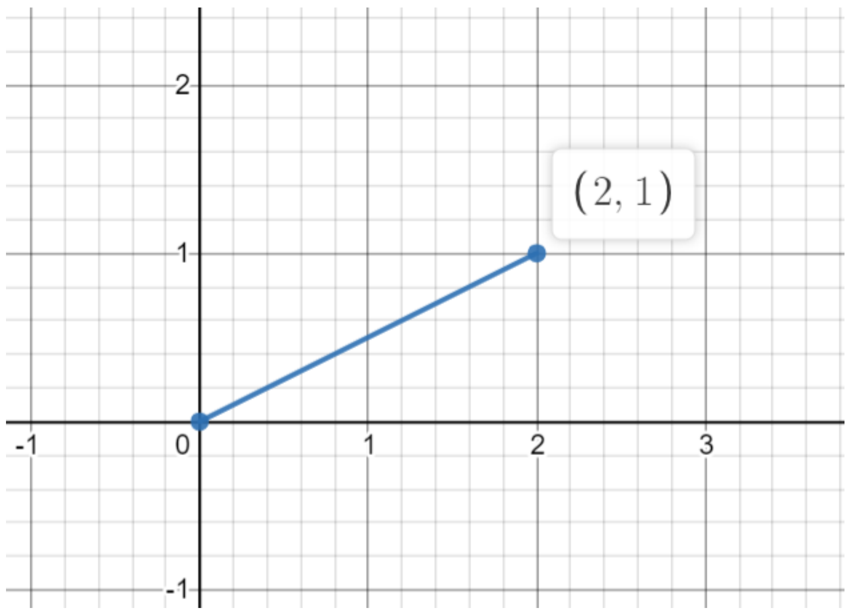
LECTURE 3: LINES

Lines

Let's look back at how we describe a line in the plane: we use the slope (read: direction) of the line and a point on the line:

$$y - y_0 = m(x - x_0)$$

The slope, $m = \frac{\text{rise}}{\text{run}}$, and notice we can encode that as a vector: $\vec{v} = \langle \text{run}, \text{rise} \rangle$ as follows:



The above vector has a slope if $\frac{1}{2}$

We can see then that any multiple of $\vec{v}$ starting at (x_0, y_0) points to a point on l. This gives us the vector equation for the line:

$\vec{l} = \vec{P}_0 + t\vec{v} \quad \vec{P}_0 = \langle x_0, y_0 \rangle \quad \vec{l} = \langle x, y \rangle$

If we use $\vec{v} = \langle l, m \rangle$, then

$\langle x, y \rangle = \vec{l} + \vec{P}_0 + t\vec{v} = \langle x_0, y_0 \rangle + \langle t, mt \rangle$

so

$$\begin{cases} x = x_0 + t \\ y = y_0 + mt \end{cases} \overset{m \neq 0}{\Longrightarrow} \begin{cases} t = x - x_0 \\ y = y_0 + mt\, t = \frac{1}{m}(y - y_0) \end{cases} \Longrightarrow \frac{1}{m}(y - y_0) = x - x_0$$
$$\Longrightarrow y - y_0 = m(x - x_0)$$

which should look familiar.

In 3 dimensions, the equation for a line looks exactly the same:

direction vector: $\vec{v} = \langle a, b, c\rangle$
position vector of point on line: $\vec{P}_0 = \langle x_0, y_0, z_0\rangle$

The vector equation of a line is then

$$\vec{l} = \vec{P}_0 + t\vec{v}$$

If we write this out:

$$\langle x, y, z\rangle = \vec{l} = \vec{P}_0 + t\vec{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

We can then separate this into 3 equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

called the parametric equations of the line.

Example 1. Find the vector and parametric equations for the line passing through $(-2, 4, 0)$ and $(1, 1, 1)$

Solution:

First, we need a direction vector for the line. If $P = (-2, 4, 0)$ and $Q = (1, 1, 1)$, a direction vector is $\vec{v} = \vec{PQ} = \langle 3, -3, 1\rangle$. So a vector equation for the line is

$$\vec{l} = \vec{0P} + t\vec{v} = \langle -2, 4, 0\rangle + t\langle 3, -3, 1\rangle$$

From this we can read off the parametric equations

$$\begin{cases} x = -2 + 3t \\ y = 4 - 3t \\ z = 0 + t \end{cases}$$

Just as above, we can combine the parametric equations

$$\begin{array}{ccc} x = x_0 + at & y = y_0 + bt & z = z_0 + ct \\ \downarrow a \neq 0 & \downarrow b \neq 0 & \downarrow c \neq 0 \\ t = \frac{x - x_0}{a} & t = \frac{y - y_0}{b} & t = \frac{z - z_0}{c} \end{array}$$

Combining these together we get the Symmetric Equations of a line:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.}$$

It could happen that one (or even 2) of the components of $\vec{v}$ are zero. An example is if $a = 0$, then the symmetric equations would take the form

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 2. Find the symmetric equations of the line in the previous example.

Solution:

$$\frac{x - (-2)}{3} = \frac{y - 4}{-3} = \frac{z - 0}{1}$$

Simplifying this we have

$$\frac{x + 2}{3} = -\left(\frac{y - 4}{3}\right) = z$$

Sometimes we don't want a whole line, but just a line segment. If we already have an equation for the whole line, we can just restrict the parameter t to start at the first point and end at the second. So you end up with something like this:

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}, \quad a \leq t \leq b.$$

The quickest way to parametrize a line segment, however, is as follows:

If we want the line segment from P to Q it's parametrized by:

$$\vec{l}(t) = (1 - t)\vec{0P} + t\vec{0Q}, \quad 0 \leq t \leq 1$$

In the plane, we know two lines are either parallel or they intersect. Lines in space, however, can be both non-parallel and non-intersecting. These are called skew lines.

Example 3. Show that the lines:

$$L_1: \; x = 3 + 2t, \; y = 4 - t, \; z = 1 + 3t$$
$$L_2: \; x = 1 + 4s, \; y = 3 - 2s, \; z = 4 + 5s$$

are skew.

Solution:

This is done in two steps. First, we show they're not parallel. This is as easy as checking if their direction vectors are parallel. The direction vectors are: $\vec{v}_1 = \langle 2, -1, 3\rangle$ for L_1 and $\vec{v}_2 = \langle 4, -2, 5\rangle$ for L_2. It's easy to see that one is not a multiple of the other, so the lines are not parallel. To see if the lines intersect, we set them equal to each other and try to solve the system:

$$\begin{cases} x = 3 + 2t = 1 + 4s \\ y = 4 - t = 3 - 2s \\ z = 1 + 3t = 4 + 5s \end{cases}$$

implies

$$\begin{cases} 2t - 4s = -2 \\ -t + 2s = -1 \\ 3t - 5s = 4 \end{cases}$$

Now the first equation is equivalent to $t - 2s = -1$ and the second is equivalent to $t - 2s = 1$ which contradict each other. Thus the system has no solution, so the lines do not intersect. Meaning, the lines are skew.

Extra Line Examples

1. Find parametric and symmetric equations for the line though the point $(1, -2, 3)$ in direction $(-1, 2, 4)$. Find another point on this line.

2. Find parametric and symmetric equations for the line though the point $(-1, 0, 4)$ in direction $(2, 1, 0)$. Find another point on this line.

3. Are these above lines parallel or skew?