

University of Notre Dame Calculus III

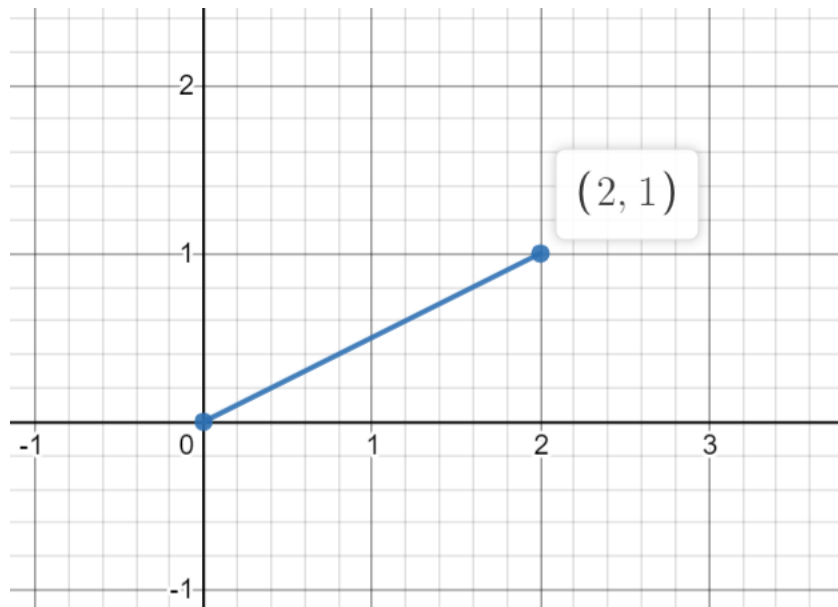
LECTURE 3: LINES

Lines

Let's look back at how we describe a line in the plane: we use the slope (read: direction) of the line and a point on the line:

$$y - y_0 = m(x - x_0)$$

The slope, $m = \frac{\text{rise}}{\text{run}}$, and notice we can encode that as a vector: $\vec{v} = \langle \text{run}, \text{rise} \rangle$ as follows:



The above vector has a slope of $\frac{1}{2}$

We can see then that any multiple of \vec{v} starting at (x_0, y_0) points to a point on l . This gives us the vector equation for the line:

$$\vec{l} = \vec{P}_0 + t\vec{v} \quad \vec{P}_0 = \langle x_0, y_0 \rangle \quad \vec{l} = \langle x, y \rangle$$

If we use $\vec{v} = \langle l, m \rangle$, then

$$\langle x, y \rangle = \vec{l} + \vec{P}_0 + t\vec{v} = \langle x_0, y_0 \rangle + \langle t, mt \rangle$$

so

$$\begin{aligned} \begin{cases} x = x_0 + t \\ y = y_0 + mt \end{cases} &\xrightarrow{m \neq 0} \begin{cases} t = x - x_0 \\ y = y_0 + m(x - x_0) \end{cases} \implies \frac{1}{m}(y - y_0) = x - x_0 \\ &\implies y - y_0 = m(x - x_0) \end{aligned}$$

which should look familiar.

In 3 dimensions, the equation for a line looks exactly the same:

$$\begin{array}{ll} \text{direction vector:} & \vec{v} = \langle a, b, c \rangle \\ \text{position vector of point on line:} & \vec{P}_0 = \langle x_0, y_0, z_0 \rangle \end{array}$$

The vector equation of a line is then

$$\vec{l} = \vec{P}_0 + t\vec{v}$$

If we write this out:

$$\langle x, y, z \rangle = \vec{l} = \vec{P}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

We can then separate this into 3 equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

called the parametric equations of the line.

Example 1. Find the vector and parametric equations for the line passing through $(-2, 4, 0)$ and $(1, 1, 1)$

Solution:

Just as above, we can combine the parametric equations

$$\begin{array}{lll} x = x_0 + at & y = y_0 + bt & z = z_0 + ct \\ \downarrow a \neq 0 & \downarrow b \neq 0 & \downarrow c \neq 0 \\ t = \frac{x-x_0}{a} & t = \frac{y-y_0}{b} & t = \frac{z-z_0}{c} \end{array}$$

Combining these together we get the Symmetric Equations of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

It could happen that one (or even 2) of the components of \vec{v} are zero. An example is if $a = 0$, then the symmetric equations would take the form

$$x = x_0, \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 2. Find the symmetric equations of the line in the previous example.

Solution:

Sometimes we don't want a whole line, but just a line segment. If we already have an equation for the whole line, we can just restrict the parameter t to start at the first point and end at the second. So you end up with something like this:

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}, \quad a \leq t \leq b.$$

The quickest way to parametrize a line segment, however, is as follows:

If we want the line segment from P to Q it's parametrized by:

$$\vec{l}(t) = (1 - t)\vec{P} + t\vec{Q}, \quad 0 \leq t \leq 1$$

In the plane, we know two lines are either parallel or they intersect. Lines in space, however, can be both non-parallel and non-intersecting. These are called skew lines.

Example 3. Show that the lines:

$$L_1 : x = 3 + 2t, y = 4 - t, z = 1 + 3t$$

$$L_2 : x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$$

are skew.

Solution:

Extra Line Examples

1. Find parametric and symmetric equations for the line through the point $(1, -2, 3)$ in direction $(-1, 2, 4)$. Find another point on this line.
2. Find parametric and symmetric equations for the line through the point $(-1, 0, 4)$ in direction $(2, 1, 0)$. Find another point on this line.
3. Are these above lines parallel or skew?