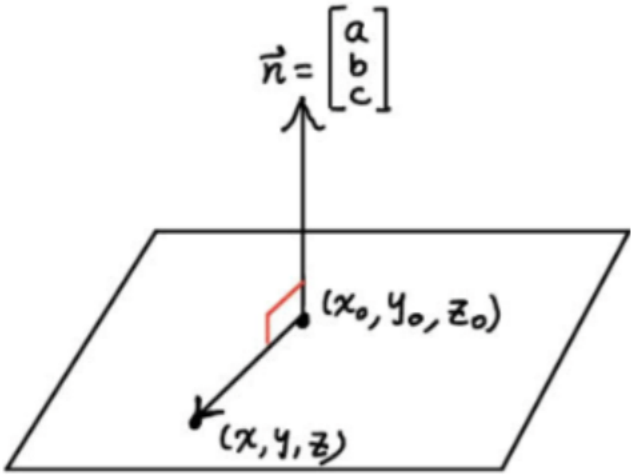
University of Notre Dame Calculus III

LECTURE 4:PLANES

Planes

The natural generalization of a line is a plane. We again need two pieces of information to get the equation of a plane:

1. A point $P_0 = (x_0, y_0, z_0)$ in the plane
2. A vector normal (perpendicular) to the plane $\vec{n} = \langle a, b, c \rangle$



$P = (x, y, z)$ is any point in the plane

How does this give us a plane?
Notice how $\vec{n} \perp \vec{P_0P}$ for any point P in the plane. So, an equation for the plane is

Vector equation of the plane Π	$\vec{n} \cdot \vec{P_0P} = 0$

Filling in $\vec{n} = \langle a, b, c \rangle$ and $\vec{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ gives the scalar equation of the plane:

$$\begin{aligned} \vec{n} \cdot \vec{P_0P} &= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Sometimes this is written as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example 1. Find an equation for the plane passing through $P = (0,1,1)$, $Q = (1,0,1)$, and $R = (1,1,0)$.

Solution:

We already have a point in the plane (3 even!), so we just need the normal vector notice we can make two vectors in the plane starting from P: $\vec{PQ}$ and $\vec{PR}$

$$\vec{PQ} = \langle 1-0, 0-1, 1-1\rangle = \langle 1, -1, 0\rangle$$
$$\vec{PR} = \langle 1-0, 1-1, 0-1\rangle = \langle 1, 0, -1\rangle$$

Now we can use these two vectors in the plane (which are not parallel!) to make a normal vector by taking their corss product:

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$$

So, an equation is:

$$\begin{aligned} \vec{n} \cdot \langle x-0, y-1, z-1\rangle &= \langle 1,1,1\rangle \cdot \langle x, y-1, z-1\rangle \\ &= x + (y-1) + (z-1) = 0 \end{aligned}$$

or equivalently

$$x + y + z = 2$$

Now, we have two kinds of objects in space: lines and planes. We already know the situation for two lines (intersecting, parallel, or skew), so how about the other pairs? Let's start with a line and a plane. Two things can happen: they're parallel or they intersect.

Example 2. Does the line

$$L: \;\; x = 3 + 3t, \;\; y = t, \;\; z = -2 + 4t$$

intersect the plane $x + y + z = 2$? If so, where?

Solution:

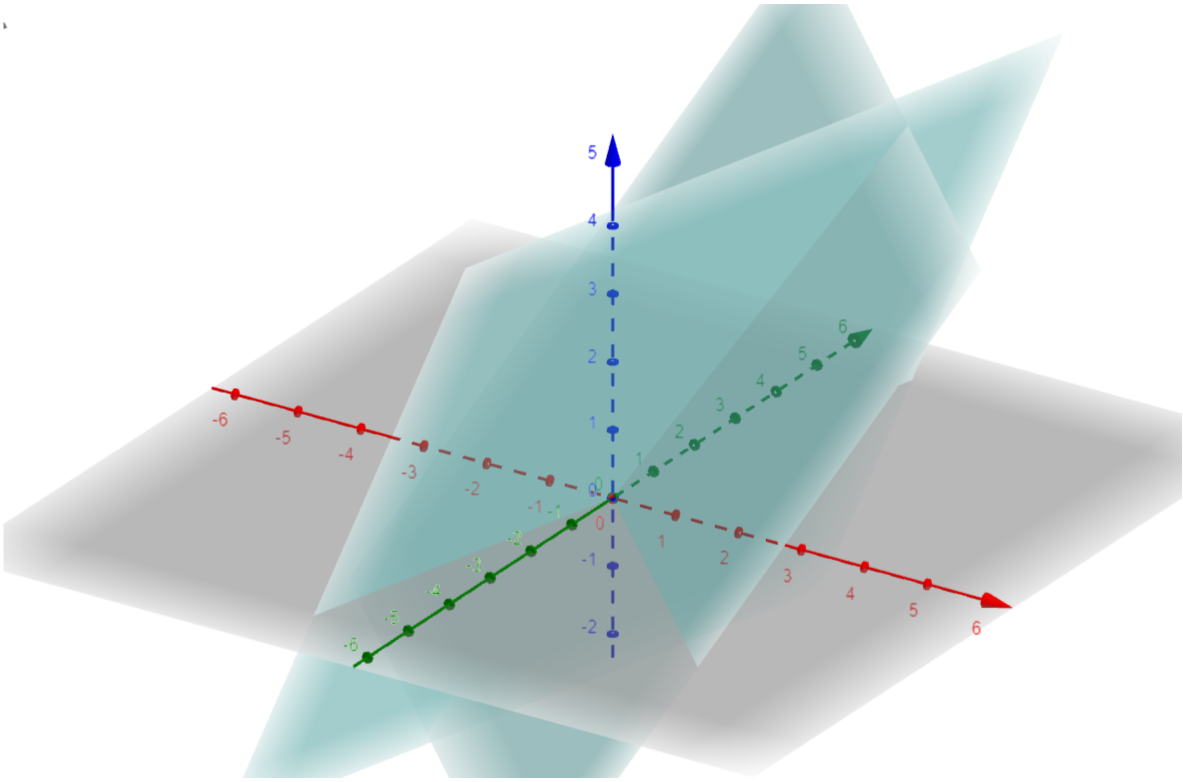
If the line intersects the plane, we can plug the line into the equation for the plane and solve for a t value.

$$x + y + z = (3 + 3t) + (t) + (-2 + 4t) = 1 + 8t = 2$$

Solving this gives $t = \frac{1}{8}$. So, they do intersect and the point of intersection* is

$$(x, y, z) = (3 + 3\left(\frac{1}{8}\right), \frac{1}{8}, -2 + 4\left(\frac{1}{8}\right)) = \left(\frac{27}{8}, \frac{1}{8}, \frac{-3}{2}\right)$$

How, now, about 2 planes? It's possible they're parallel (to check this, check if their normal vectors are parallel). More likely, though, they'll intersect. As you can probably see, they don't intersect in a point, but a line!



Example 3. Do the planes $2x-3y+4z=5$ and $x+6y+4z=3$ intersect? If so, what is the angle of their intersection*? Also, give an equation for their line of intersection*.

Solution:

The normal vectors of the planes are

$$\vec{n}_1=\langle 2,-3,4\rangle \qquad \vec{n}_2=\langle 1,6,4\rangle$$

which can easily be seen to not be parallel since one is not a multiple of the other. So the planes are not parallel, thus they intersect. The angle of intersection* is the same as the angle between their normal vectors:

$$\theta=\arccos\left(\frac{\vec{n}_1\cdot\vec{n}_2}{||\vec{n}_1||\,||\vec{n}_1||}\right)=\arccos\left(\frac{(2)(1)+(-3)(6)+(4)(4)}{(\sqrt{4+9+16})(\sqrt{1+36+16})}\right)=\arccos(0)=\frac{\pi}{2}$$

(This actually means the planes are perpendicular!)
Now, for the line of intersection*, we need a point and a direction vector. Let's start with the direction. The line lies in both planes, so its direction vector $\vec{v}$ must be perpendicular to both $\vec{n}_1$ and $\vec{n}_2$ since its parallel to both planes. We have a trick for creating a vector orthogonal to two given vectors: the cross product.

$$\begin{aligned}\vec{v}=\vec{n}_1\times\vec{n}_2&=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ 2&-3&4\\ 1&6&4\end{vmatrix}\\ &=\langle -12-24,-(8-4),12-(-3)\rangle\\ &=\langle -36,-4,15\rangle\end{aligned}$$

We may as well choose $\vec{v}=\vec{n}_1\times\vec{n}_2$. Now, for a point on the line, we just need to find a point on both planes, that is, a solution to both $2x-3y+4z=5$ and $x+6y+4z=3$. We have two equations and three variables, so we'll have to choose a avalue for one of them, say $z=0$. Then, we need to solve the system:

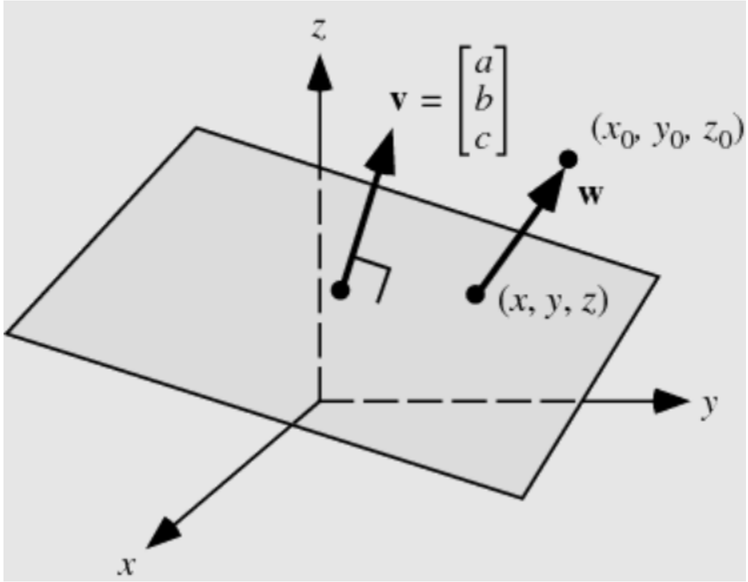
$$\begin{cases}2x-3y=5\\ x+6y=3\end{cases}$$

Two times the first plus the second yields: $5x=13$, so $x=\frac{13}{5}$. So plugging this back into the second we have $6y=3-\frac{13}{5}=\frac{2}{5}$, so $y=\frac{1}{15}$. This means the point $(\frac{13}{5},\frac{1}{15},0)$ is on the line.
The symmetric equations for this line then are

$$\frac{x-\frac{13}{5}}{-36}=\frac{y-\frac{1}{15}}{-4}=\frac{z}{15}$$

Consider the following situation:

We're given a plane Π and a point P. How can we find the distance, D, from the plane to the point?



First, we know that the shortest path from the plane to the point is a straight line perpendicular to the plane, that is a line in the direction of $\vec{n}$, the normal vector to Π. Notice that if we take some point P_0 on Π and connect it to P, we get a vector connecting Π to P, and, moreover, if we project $\overrightarrow{P_0P_1}$ onto $\vec{n}$, we get a vector perpendicular to Π which starts on Π and ends at P. The length of this vector, then, is precisely D, i.e.

$$\boxed{D = ||\text{proj}_{\vec{n}} \overrightarrow{P_0P_1}|| = |\text{comp}_{\vec{n}} \overrightarrow{P_0P_1}|}$$

If $\vec{n} = \langle a, b, c \rangle$, $P_0 = (x_0, y_0, z_0)$, and $P_1 = (x_1, y_1, z_1)$, then

$$D = |\text{comp}_{\vec{n}} \overrightarrow{P_0P_1}| = \frac{|\vec{n} \cdot \overrightarrow{P_0P_1}|}{||\vec{n}||} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

If the plane is written as $ax + by + cz + d = 0$ then

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Let's see how this can be used to answer a related question.

Example 4. Find the distance between the parallel planes $x-4y+2z=0$ and $2x-8y+4z=-1$

Solution:

Our situation looks as follows:

If we forgot everything except P_1 from the top plane, we've reduced the problem to the distance between a point and a plane. First, we need to find a P_1 (it doesn't matter which plane P_1 is on, as long as P_0 is on the other one). Let's take P_1 on the second plane. Any point works, so the easiest way to get one is to make two components equal to zero, e.g., take $P_1 = (-\frac{1}{2}, 0, 0)$. A point on the other plane is $P_0 = (0,0,0)$. A normal vector to the planes is $\vec{n} = \langle 1, -4, 2\rangle$, so

$$D = |\text{comp}_{\vec{n}} \vec{P_0P_1}| = \frac{|\vec{n} \cdot \vec{P_0P_1}|}{||\vec{n}||} = \frac{|(1)\left(-\frac{1}{2}\right) + (-4)(0) + (2)(0)|}{\sqrt{1+16+4}} = \frac{\frac{1}{2}}{\sqrt{21}} = \frac{1}{2\sqrt{21}}$$

Extra Plane and Line Examples

1. Find the equation of the plane through the points $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$.
2. Find the equation of the plane through the point $(3,-2,8)$ and parallel to the plane $z = x+y$.
3. Are the planes given by the equations $9x-3y+6z=2$ resp. $2y=6x+4z$ parallel, perpendicular or neither? If neither, find the angle between them.
4. Find the distance from the point $(-6,3,5)$ to the plane $x-2y-4z=8$.