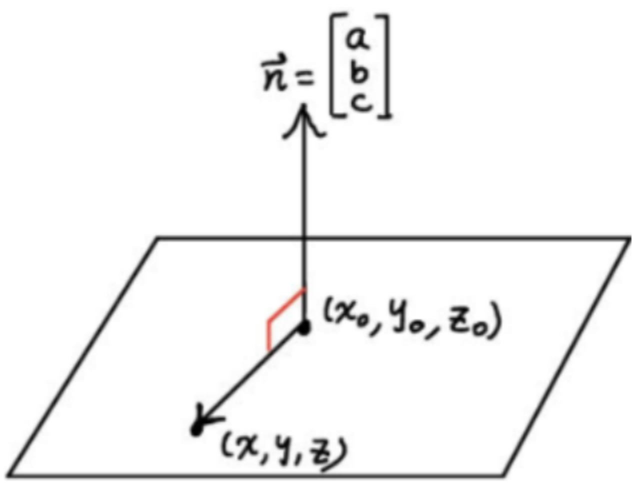
University of Notre Dame Calculus III

LECTURE 4:PLANES

Planes

The natural generalization of a line is a plane. We again need two pieces of information to get the equation of a plane:

1. A point $P_0 = (x_0, y_0, z_0)$ in the plane
2. A vector normal (perpendicular) to the plane $\vec{n} = \langle a, b, c \rangle$



$P = (x, y, z)$ is any point in the plane

How does this give us a plane?
Notice how $\vec{n} \perp \vec{P_0P}$ for any point P in the plane. So, an equation for the plane is

Vector equation of the plane Π	$\vec{n} \cdot \vec{P_0P} = 0$

Filling in $\vec{n} = \langle a, b, c \rangle$ and $\vec{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ gives the scalar equation of the plane:

$$\begin{aligned} \vec{n} \cdot \vec{P_0P} &= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Sometimes this is written as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example 1. Find an equation for the plane passing through $P = (0,1,1)$, $Q = (1,0,1)$, and $R = (1,1,0)$.

Solution:

Now, we have two kinds of objects in space: lines and planes. We already know the situation for two lines (intersecting, parallel, or skew), so how about the other pairs? Let's start with a line and a plane. Two things can happen: they're parallel or they intersect.

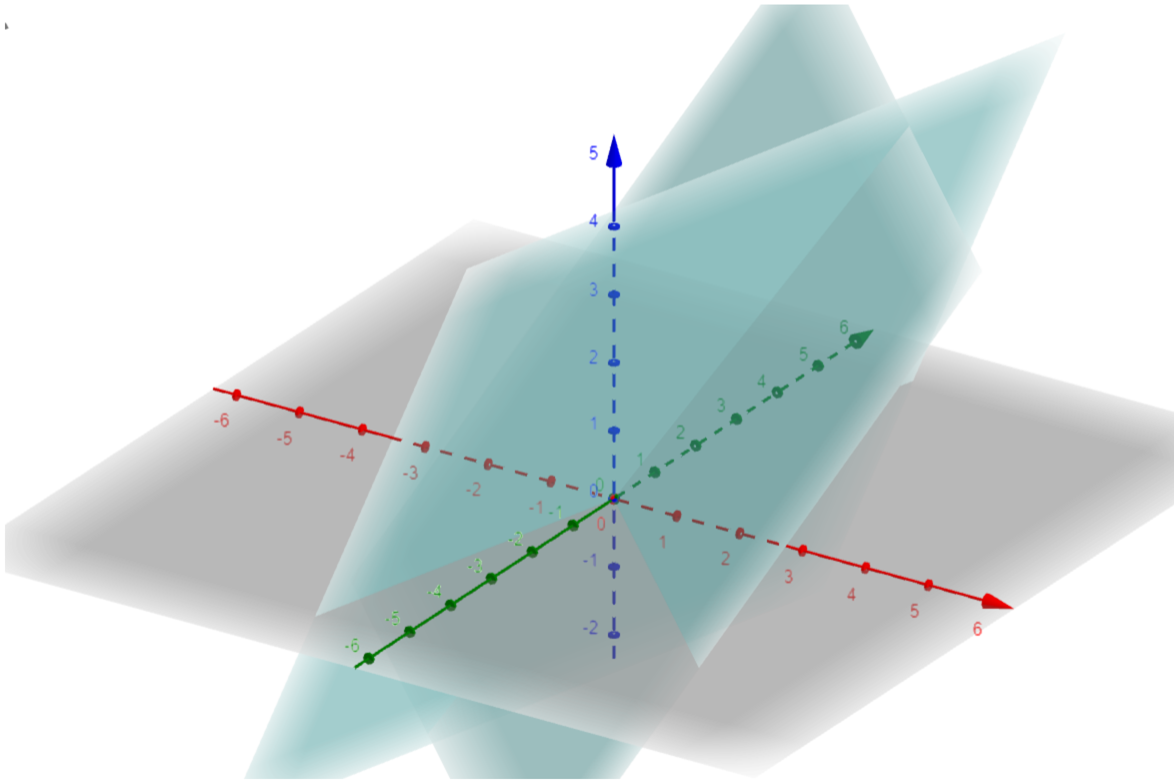
Example 2. Does the line

$$L: \quad x = 3 + 3t, \quad y = t, \quad z = -2 + 4t$$

intersect the plane $x + y + z = 2$? If so, where?

Solution:

How, now, about 2 planes? It's possible they're parallel (to check this, check if their normal vectors are parallel). More likely, though, they'll intersect. As you can probably see, they don't intersect in a point, but a line!

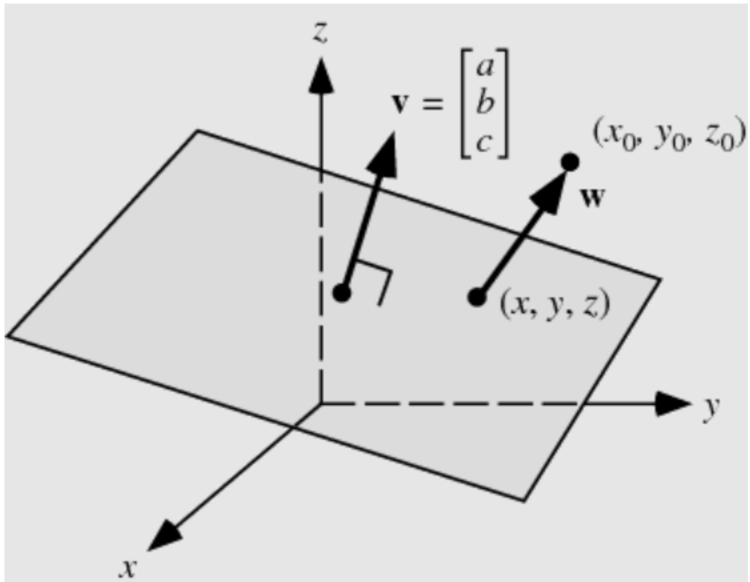


Example 3. Do the planes $2x - 3y + 4z = 5$ and $x + 6y + 4z = 3$ intersect? If so, what is the angle of their intersection*? Also, give an equation for their line of intersection*.

Solution:

Consider the following situation:

We're given a plane Π and a point P. How can we find the distance, D, from the plane to the point?



First, we know that the shortest path from the plane to the point is a straight line perpendicular to the plane, that is a line in the direction of $\vec{n}$, the normal vector to Π. Notice that if we take some point P_0 on Π and connect it to P, we get a vector connecting Π to P, and, moreover, if we project $\vec{P_0P_1}$ onto $\vec{n}$, we get a vector perpendicular to Π which starts on Π and ends at P. The length of this vector, then, is precisely D, i.e.

$$D = ||\text{proj}_{\vec{n}} \vec{P_0P_1}|| = |\text{comp}_{\vec{n}} \vec{P_0P_1}|$$

If $\vec{n} = \langle a, b, c\rangle$, $P_0 = (x_0, y_0, z_0)$, and $P_1 = (x_1, y_1, z_1)$, then

$$D = |\text{comp}_{\vec{n}} \vec{P_0P_1}| = \frac{|\vec{n} \cdot \vec{P_0P_1}|}{||\vec{n}||} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

If the plane is written as $ax + by + cz + d = 0$ then

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Let's see how this can be used to answer a related question.

Example 4. Find the distance between the parallel planes $x-4y+2z=0$ and $2x-8y+4z=-1$

Solution:

Extra Plane and Line Examples

1. Find the equation of the plane through the points $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$.
2. Find the equation of the plane through the point $(3,-2,8)$ and parallel to the plane $z=x+y$.
3. Are the planes given by the equations $9x-3y+6z=2$ resp. $2y=6x+4z$ parallel, perpendicular or neither? If neither, find the angle between them.
4. Find the distance from the point $(-6,3,5)$ to the plane $x-2y-4z=8$.