University of Notre Dame Calculus III

LECTURE 6: SPACE CURVES

Vector Valued Functions and Space Curves

A vector-valued function is a function whose output is a vector. We have already encountered one: the vector equation for a line

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}$$

More generally, they will have the form

$$\begin{aligned}\vec{r}(t) &= \langle f(t), g(t), h(t)\rangle \\ &= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}\end{aligned}$$

The input variable (in this case it's t) is called the parameter. Since $\vec{r}$ is a function, we can ask about its domain. The domain of a vector-valued function is the "intersection*" of the domains of its components functions, that is, the values common to the domains of each of f, g, and h.

Example 1. What is the domain of $\vec{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1)\rangle$?

Solution:

First, we find the domains of each of the component functions:

function	$f(t) = \sqrt{4-t^2}$	$g(t) = e^{-3t}$	$h(t) = \ln(t+1)$
domain	$-2 \le t \le 2$	$-\infty < t < \infty$	$-1 < t < \infty$

The t-values in common to each of these are $-1 < t \le 2$. So the domain is $(-1, 2]$.

As with normal functions, we can take limits of vector-valued functions:
If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

And this leads us to the definition of continuity for vector valued functions:

Definition 1. *A vector valued function $\vec{r}(t)$ is called continuous at a if*

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

Let's look at some examples of vector-valued functions.

Example 2.

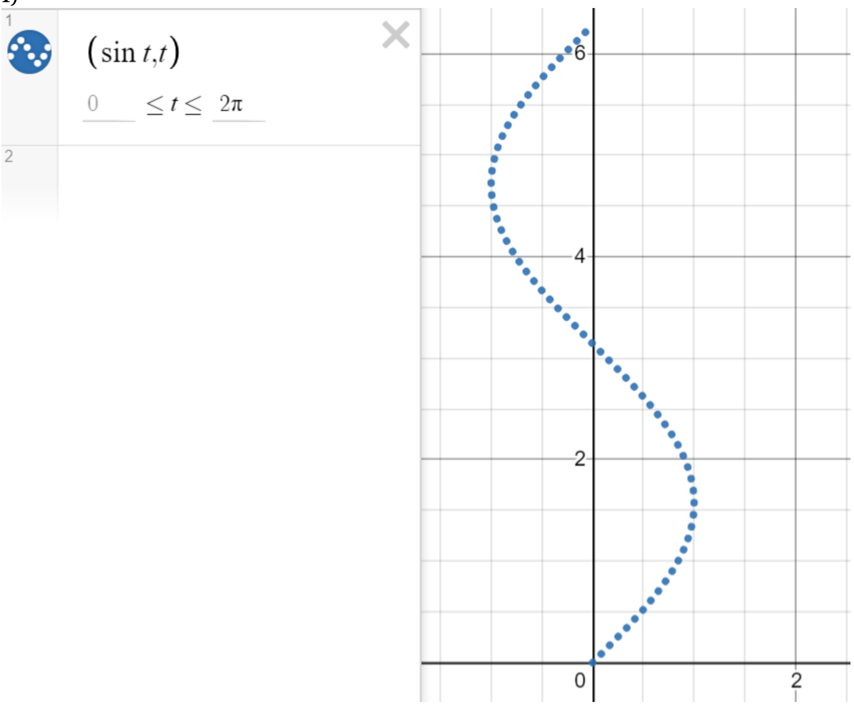
i) $\vec{r}(t) = \langle \sin t, t \rangle$ ii) $\vec{r}(t) = \langle \sqrt{2}\cos t, \sin t \rangle$

iii) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ iv) $\langle t, \sin t, 2\cos t \rangle$

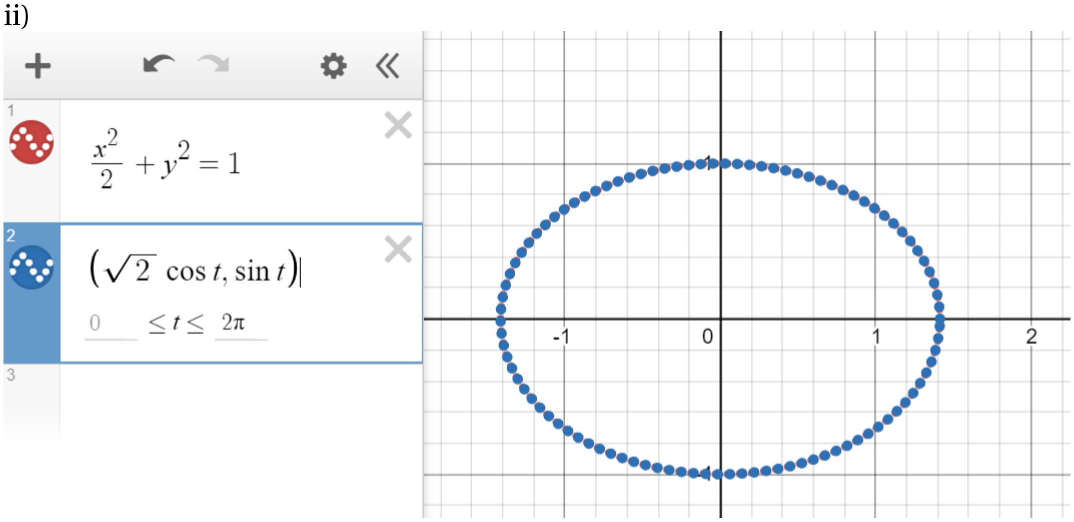
v) $\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$

Solution:

i)



ii)

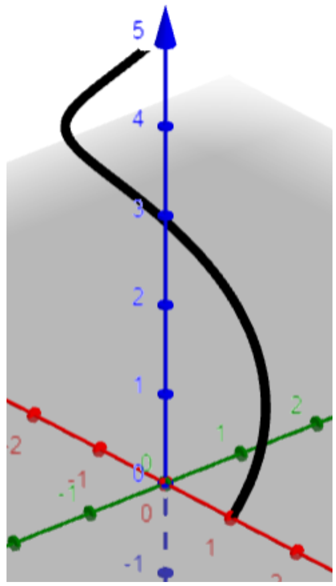


Notice that $x = \sqrt{2}\cos t$ and $y = \sin t$ satisfies $\frac{x^2}{2} + y^2 = 1$ the equation of an ellipse!

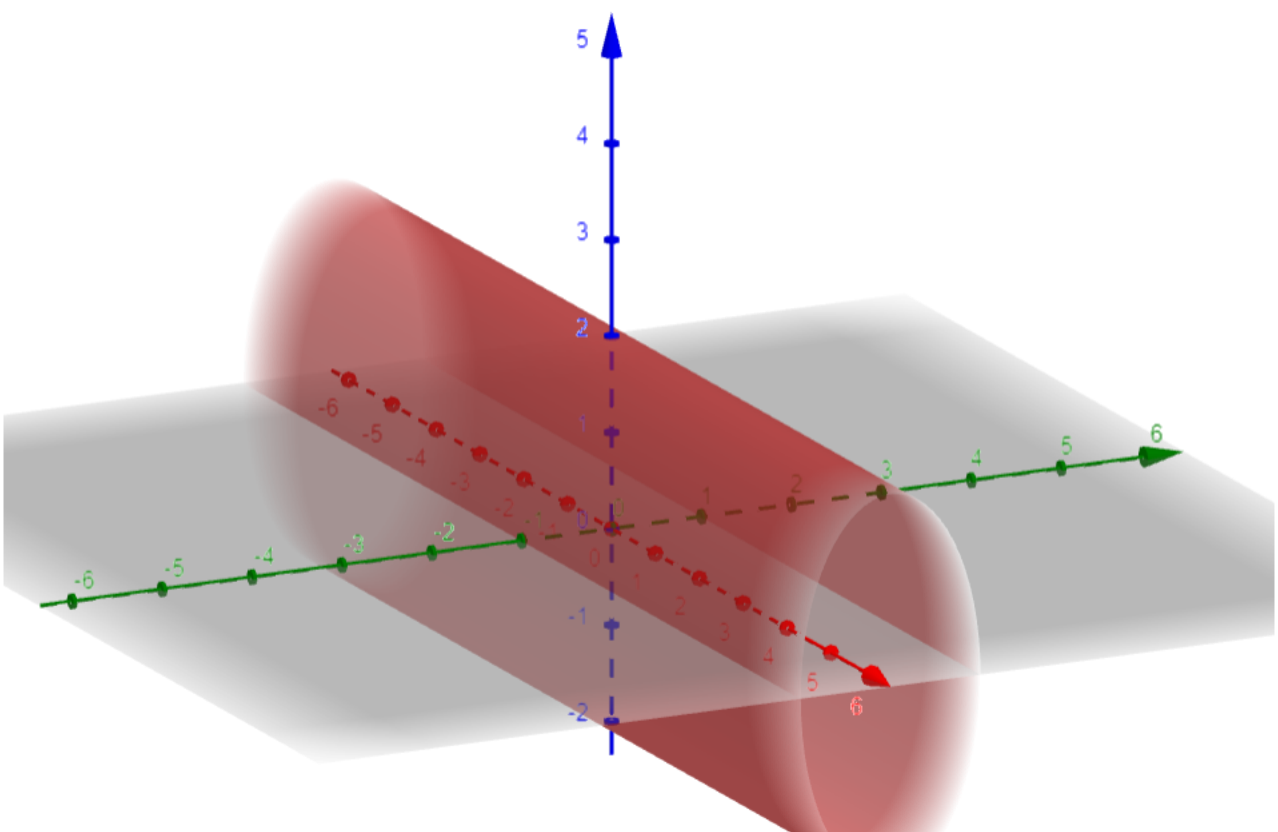
iii) With 3D space curves, it's often useful to find a surface that your curve sits on. In this case we have

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle \cos t, \sin t, t\rangle$$

so $[x(t)]^2 + [y(t)]^2 = 1$. This means the curve sits on the cylinder and climbs up it as t increases:



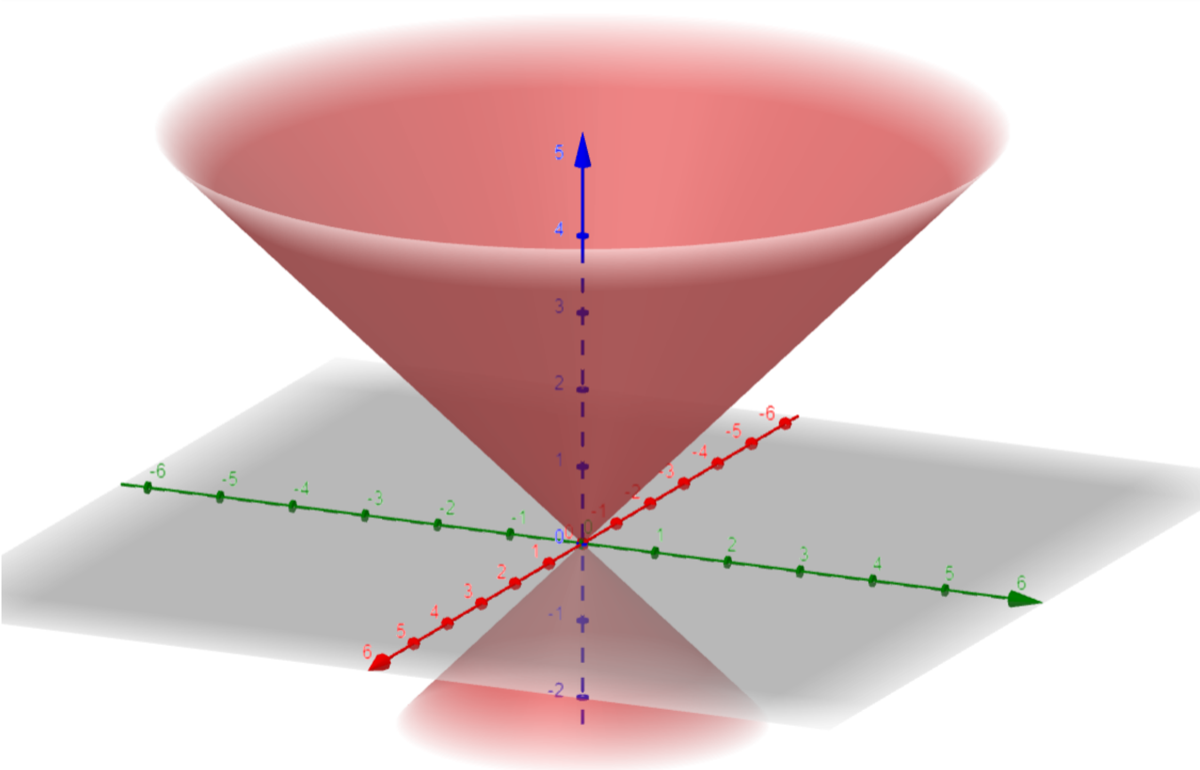
iv) Here, $[y(t)]^2 + \frac{[z(t)]^2}{4} = 1$, so the curve sits on the elliptic cylinder $y^2 + \frac{z^2}{4} = 1$ (it opens along the x-axis).



v) This one satisfies

$$[x(t)]^2 + [y(t)]^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = [z(t)]^2$$

meaning it sits on the cone $x^2 + y^2 = z^2$



Often of importance is the intersection* of surfaces. The result is generically a curve (i.e., the intersection* of two planes is a line, as we know). It is important to be able to quantify these "curves of intersection*" by parameterizing them (finding a vector-valued function whose image is the curve).

Example 3. Find a vector function representing the intersection* of $z = x^2$ and $x^2 + y^2 = 4$.

Solution:

We know that whatever the vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is, it must satisfy the equations of the surfaces since it lies on both of them, that is, we must have

$$z(t) = [x(t)]^2 \qquad \text{and} \qquad [x(t)]^2 + [y(t)]^2 = 4\ .$$

The second equation suggests taking

$$x(t) = 2\cos t \qquad \text{and} \qquad y(t) = 2\sin t$$

and first then gives $z(t) = 4\cos^2 t$.
Thus the vector equation is

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\cos^2 t\rangle\ .$$

Extra Examples

1. Sketch the curve given by $\mathbf{r}(t) = \langle \sin 2\pi t, t, \cos 2\pi t\rangle$, indicating by an arrow the direction in which t increases.

2. What is the domain of the vector function $\mathbf{r}(t) = \langle \frac{1}{t+2}, \cos t, \ln(16 - t^2)\rangle$?

3. Determine the parametric equations and the vector equation for the intersection* of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.