University of Notre Dame Calculus III

LECTURE 7:CALCULUS OF SPACE CURVES

Derivatives and Integrals

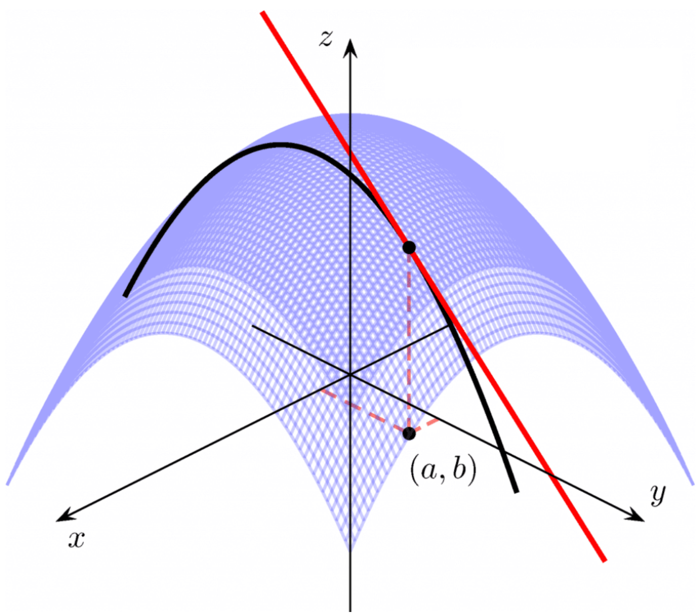
What were the main applications of limits back in Calc I? Differentiation and integration!

Derivatives

Definition 1. *The derivative of a vector valued function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is*

$$\begin{aligned}\frac{d\vec{r}}{dt} = \vec{r}\,'(t) &= \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}\\ &= \langle f'(t), g'(t), h'(t)\rangle\end{aligned}$$

Recall that the derivative of a function gave us the slope of its tangent line. For vector-valued functions, the derivative gives us a tangent vector (pointing in the direction of increasing t-values). This vector can be used as a direction vector for the tangent line.



$\vec{r}\,'(a)$ is the tangent vector to $\vec{r}(t)$ at $t = a$ (provided $\vec{r}\,'(a) \neq \vec{0}$). A very important variation of the tangent vector is the unit tangent vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$$

*This will be important in the next section.

Example 1. Let $\vec{r}(t) = \langle t\cos t, t, t\sin t\rangle$. Find:
i) $\vec{r}'(t)$
ii) $\vec{T}(t)$ iii) an equation for the tangent line to $\vec{r}(t)$ at $t = \pi$.

Solution:

i) $\vec{r}'(t) = \langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$
ii) First we need $||\vec{r}'(t)||$:

$$\begin{aligned}
||\vec{r}'(t)|| &= \sqrt{(\cos t - t\sin t)^2 + 1 + (\sin t + t\sin t)^2} \\
&= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 + 1 + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} \\
&= \sqrt{1 + t^2 + 1} = \sqrt{t^2 + 2}
\end{aligned}$$

So,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{1}{\sqrt{t^2+2}}\langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$$

iii) A direction vector for the tangent line at $t = \pi$ is

$$\vec{r}'(\pi) = \langle \cos\pi - \pi\sin\pi, 1, \sin\pi + \pi\cos\pi\rangle = \langle -1, 1, -\pi\rangle$$

(note that we could have also used $\vec{T}(\pi)$)
A point on the tangent line is $\vec{r}(\pi) = \langle -\pi, \pi, 0\rangle$, so an equation for the tangent line is

$$\vec{l}(s) = \vec{r}(\pi) + s\vec{r}'(\pi) = \langle -\pi - s, \pi + s, \pi s\rangle$$

Properties of Derivatives

Let $\vec{u}(t)$, $\vec{v}(t)$ be vector functions, c a constant, and $f(t)$ a scalar function. Then

1. $\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}'(t)$
4. $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)$
5. $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$
6. $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

The first two of these establish the linearity of the derivative of vector functions. The next three are product rules of vector functions. The final is the chain rule of vector functions. There is a useful consequence of #4. Suppose $\vec{r}'(t) \neq \vec{0}$ then

$$\begin{aligned}\frac{d}{dt}||\vec{r}(t)|| = \frac{d}{dt}\sqrt{\vec{r}(t)\cdot\vec{r}(t)} &= \frac{1}{2}\frac{\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)]}{\sqrt{\vec{r}(t)\cdot\vec{r}(t)}} \\ &= \frac{1}{2}\frac{2\vec{r}(t)\cdot\vec{r}'(t)}{||\vec{r}(t)||} = \frac{\vec{r}(t)\cdot\vec{r}'(t)}{||\vec{r}(t)||}\end{aligned}$$

Integration

Definition 2. *The definite integral of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is*

$$\begin{aligned}\int_a^b \vec{r}(t)\,d &= \lim_{n\to\infty}\sum_{i=1}^{n} \vec{r}(t_i^*)\Delta t_i \\ &= \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle\end{aligned}$$

The t_i^* are chosen from the i^{th} piece of a partition of $[a,b]$ into n pieces. One can also define indefinite integrals.

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\hat{i} + \left(\int g(t)\,dt\right)\hat{j} + \left(\int h(t)\,dt\right)\hat{k}$$

Example 2. Find $\int \vec{r}(t)\,dt$ and $\int_0^2 \vec{r}(t)\,dt$ where $\vec{r}(t) = t\hat{i} - t^3\hat{k}$.

Solution:

$$\begin{aligned}\int \vec{r}(t)\,dt &= \left(\int t\,dt\right)\hat{i} + \left(\int 0\,dt\right)\hat{j} + \left(\int -t^3\,dt\right)\hat{k} \\ &= \left(\frac{1}{2}t^2 + C_1\right)\hat{i} + C_2\hat{j} + \left(-\frac{1}{4}t^4 + C_3\right)\hat{k}\end{aligned}$$

$$\int_0^2 \vec{r}(t)\,dt = \left(\frac{1}{2}t^2\Big|_0^2\right)\hat{i} + 0\Big|_0^2\hat{j} + \left(-\frac{1}{4}t^4\Big|_0^2\right)\hat{k} = 2\hat{i} - 4\hat{k}$$

Application

Suppose two curves $\mathbf{r_1}$ and $\mathbf{r_2}$ intersect at a point P. Then the angle they intersect at can be determined by finding the angle of intersection* of the tangent vectors at the point P.

Example 3. Suppose $\mathbf{r_1}(t) = \langle \cos(t), -\sin(t), t\rangle$ and $\mathbf{r_2}(s) = \langle -s, s^2 - 1, \ln(s) + \pi\rangle$. Then what is the angle of intersection* at the point $P = (-1, 0, \pi)$?

Solution:

From the z component, we can clearly see that t needs to be π at the point of intersection* and that s needs to be 1. So we next need to find tangent vectors.

$$\mathbf{r}_1'(t) = \langle -\sin(t), -\cos(t), 1\rangle \mid_{t=\pi} = \langle 0, 1, 1\rangle$$

$$\mathbf{r}_2'(t) = \langle -1, 2s, \frac{1}{s}\rangle \mid_{s=1} = \langle -1, 2, 1\rangle$$

Thus if θ is the angle of intersection*, we can find it using the dot product formula

$$\langle 0, -1, 1\rangle \cdot \langle -1, 2, 1\rangle = ||\langle 0, -1, 1\rangle|| ||\langle -1, 2, 1\rangle|| \cos(\theta).$$

Which means

$$2 + 1 = \sqrt{1+1}\sqrt{1+4+1}\cos(\theta)$$

Therefore

$$\theta = \frac{\pi}{6}.$$

Extra Examples

1. Find the tangent vector and the unit tangent vector of the space curve $\mathbf{r}(t) = \langle \cos t, 3t, 2\sin 2t\rangle$ for $t = 0$.

2. Show that if $|\mathbf{r}(t)|$ is a constant c, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.