

University of Notre Dame Calculus III

LECTURE 8:

Arc Length and Curvature

In Calc II we found the arc length of a plane curve $x(t) = f(t)$, $y(t) = g(t)$, $a \leq t \leq b$ as

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_a^b \sqrt{dx^2 + dy^2} \end{aligned}$$

This was done by approximating the curve by straight lines. We can do the same thing for curves in \mathbb{R}^3 . This leads to:

Definition 1. The arc length of the curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

There are some technical assumptions to this formula:

- $\vec{r}(t)$ does not cross itself between $t = a$ and $t = b$
- f' , g' , and h' must be continuous (i.e. \vec{r} is C^1)

Notice that because

$$\|\vec{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

we have that the arc length can be computed as

$$L = \int_a^b \|\vec{r}'(t)\| dt$$

Example 1. Find the arc length of

$$\vec{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$$

where $-5 \leq t \leq 5$.

Solution:

$$\begin{aligned} \vec{r}'(t) &= \langle 1, -3 \sin t, 3 \cos t \rangle \|\vec{r}'(t)\| \\ &= \sqrt{1 + 9 \sin^2 t + 9 \cos^2 t} \\ &= \sqrt{10} \end{aligned}$$

$$\text{So } L = \int_{-5}^5 \sqrt{10} dt = 10\sqrt{10}$$

A curve C need not have a unique representation by a vector function, in fact, none do. For example $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \leq t \leq 2$ is also represented by $\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \leq u \leq 2$. \vec{r}_1 and \vec{r}_2 are called

parametrizations of C . There is one particular parametrization we care about, and it is found as follows: Suppose C is given by $\vec{r}(t)$, $a \leq t \leq b$, with \vec{r}' continuous and $\vec{r}(t)$ traverses C exactly once. We can define the arc length function

$$s(t) = \int_a^t \|\vec{r}'(u)\| du$$

which tells us the distance traveled at time t . Now, suppose we can solve this equation for t in terms of s . In other words invert $s(t)$ so that $t = t(s)$. Then:

Definition 2. The Arc Length Reparametrization of $\vec{r}(t)$ is

$$\vec{r} = \vec{r}(t(s)), \quad 0 \leq s \leq L$$

where $L = \text{arc length of } \vec{r} \text{ from } t = a \text{ to } t = b$.

Example 2. Reparametrize $\vec{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$, $-5 \leq t \leq 5$ with respect to arc length.

Solution:

Firs, find the arc length function:

$$\begin{aligned} s = s(t) &= \int_{-5}^t \|\vec{r}'(u)\| du \\ &= \int_{-5}^t \sqrt{10} du \\ &= \sqrt{10}t + 5\sqrt{10} \end{aligned}$$

Solving for t gives: $t = \frac{s - 5\sqrt{10}}{\sqrt{10}} = t(s)$.

From the last example, $L = 10\sqrt{10}$, so the bounds on s are $0 \leq s \leq 10\sqrt{10}$, so the bounds on s , thus the reparametrization is:

$$\vec{r}(t(s)) = \left\langle \frac{s - 5\sqrt{10}}{\sqrt{10}}, 3 \cos\left(\frac{s - 5\sqrt{10}}{\sqrt{10}}\right), 3 \sin\left(\frac{s - 5\sqrt{10}}{\sqrt{10}}\right) \right\rangle, \quad 0 \leq s \leq 10\sqrt{10}.$$

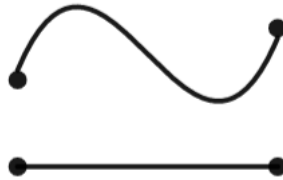
An interesting fact about arc length reparametrizations:

$$\frac{d\vec{r}}{ds} = \frac{d}{ds}(\vec{r}(t(s))) = \left(\frac{dt}{ds}\right) \vec{r}'(t(s)) = \frac{1}{\|\vec{r}'(t)\|} \vec{r}'(t(s))$$

So, $\left\| \frac{d\vec{r}}{ds} \right\| = 1$, that is, arc length reparametrizations always move with unit speed!

Curvature

Intuitively, curvature is a measure of how sharply a curve bends. Pictorially, the curved line has larger curvature than the bottom line does:



Definition 3. A parametrization $\vec{r}(t)$ is called smooth on an interval I if \vec{r}' is continuous on I and $\vec{r}'(t) \neq \vec{0}$ for any $t \in I$. A curve C is called smooth if it has a smooth parametrization.

We quantify curvature as the rate of change of the unit tangent vector with respect to arc length. In symbols, the curvature of \vec{r} is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

Now $\frac{d\vec{T}}{ds}$ can often be messy to compute, however, we have a trick: by the chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \frac{ds}{dt} = \frac{d\vec{T}}{ds} \|\vec{r}'(t)\|$$

So, a more convenient formula for curvature is

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

Example 3. Find the curvature of a circle of radius a .

Solution:

Parametrize it: $\vec{r}(t) = \langle -a \sin t, a \cos t \rangle$ then $\vec{r}'(t) = \langle -a \cos t, -a \sin t \rangle$, $\|\vec{r}'(t)\| = a$. So,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle -\cos t, -\sin t \rangle$$

Then $\vec{T}'(t) = \langle \sin t, -\cos t \rangle$ and $\|\vec{T}'(t)\| = 1$, so

$$\kappa(t) = \frac{1}{a}.$$

Even that formula is more effort than needed. Another is

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

Example 4. Find the curvature of $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ at $(0, 1, 1)$.

Solution:

First, find the relevant t value. Since $\vec{r}(0) = \langle 0, e^0, e^{-0} \rangle = \langle 0, 1, 1 \rangle$, the t value is $t = 0$. We need $\vec{r}'(t)$ and $\vec{r}''(t)$:

$$\vec{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle \quad \vec{r}''(t) = \langle 0, e^t, e^{-t} \rangle$$

Now, we could either plug all of this into the formula for $\kappa(t)$ first, then plug in $t = 0$, or plug in $t = 0$ now, then compute $\kappa(0)$. We'll do the latter.

$$\vec{r}'(0) = \langle \sqrt{2}, 1, -1 \rangle \quad \vec{r}''(0) = \langle 0, 1, 1 \rangle$$

So

$$\vec{r}'(0) \times \vec{r}''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2} & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2, -\sqrt{2}, \sqrt{2} \rangle.$$

Then $\|\vec{r}'(0)\| = \sqrt{2 + 1 + 1} = \sqrt{4}$ and $\|\vec{r}'(0) \times \vec{r}''(0)\| = \sqrt{4 + 2 + 2} = \sqrt{8} = 2\sqrt{2}$. So

$$\kappa(0) = \frac{2\sqrt{2}}{2^3} = \frac{\sqrt{2}}{4}$$

In the special case of a plane curve $y = f(x)$, by parametrizing it as $\vec{r}(x) = \langle x, f(x) \rangle$ we get

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}.$$

Frenet-Serret Frame "T - N - B Frame"

This consists of 3 vectors derived from a parametrization, $\vec{r}(t)$: $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$. We already know one of them, the other two are

Unit Normal Vector (Requiring $\|\vec{T}'(t)\| \neq 0$, equivalently $\kappa(t) \neq 0$)

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Binormal Vector

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Since $\|\vec{T}(t)\| = 1$, we have $\vec{T}(t) \cdot \vec{T}'(t) = 0$, so $\vec{T} \perp \vec{N}$. By definition of \times , $\vec{B} \perp \vec{T}$, \vec{N} , so the three vectors are all orthogonal to each other. Thus since $\|\vec{T}\| = \|\vec{N}\| = 1$, we have $\|\vec{B}\| = \|\vec{T} \times \vec{N}\| = \|\vec{T}\| \|\vec{N}\| \sin \frac{\pi}{2} = 1$, thus all of \vec{T} , \vec{N} , and \vec{B} and hard to compute, so here's an alternate way:

$$\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|} \quad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t).$$

\vec{N} always points in the direction the curve is bending and \vec{B} points orthogonal to the motion of the curve.

We can create some planes using \vec{T} , \vec{N} , and \vec{B} .

Normal Plane: This plane is perpendicular to $\vec{r}(t)$. It is determined by \vec{N} and \vec{B} , and so has \vec{T} as a vector orthogonal to it.

Osculating Plane: This plane best captures the motion of the curve. It is determined by \vec{T} and \vec{N} , and so has \vec{B} as a vector perpendicular to it.

Rectifying Plane: This plane determined by \vec{T} and \vec{B} . We won't bother with this one.

Example 5. Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$ and find equations for the normal and osculating planes at $(\pi/2, 0, 3)$.

Solution:

Let's begin with computing $\vec{T}(t)$: $\vec{r}'(t) = \langle 1, -3 \sin t, 3 \cos t \rangle$, $\|\vec{r}'(t)\| = \sqrt{1 + 9 \sin^2 t + 9 \cos^2 t} = \sqrt{10}$. So

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{10}} \langle 1, -3 \sin t, 3 \cos t \rangle.$$

This is pretty tame, so we can differentiate it:

$$\vec{T}'(t) = \frac{1}{\sqrt{10}} \langle 0, -3 \cos t, -3 \sin t \rangle$$

So

$$\|\vec{T}'(t)\| = \sqrt{0^2 + \frac{9 \cos^2 t}{10} + \frac{9 \sin^2 t}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

Thus,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\frac{1}{\sqrt{10}} \langle 0, -3 \cos t, -3 \sin t \rangle}{\frac{3}{\sqrt{10}}} = \langle 0, -\cos t, -\sin t \rangle$$

Finally

$$\begin{aligned} \vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \sin t & \frac{3}{\sqrt{10}} \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} \\ &= \left\langle \frac{3}{\sqrt{10}} \sin^2 t + \frac{3}{\sqrt{10}} \cos^2 t, \frac{1}{\sqrt{10}} \sin t, -\frac{1}{\sqrt{10}} \cos t \right\rangle \\ &= \frac{1}{\sqrt{10}} \langle 3, \sin t, -\cos t \rangle \end{aligned}$$

The t -value corresponding to $(\pi/2, 0, 3)$ is $t = \pi/2$. So, we have:

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, 0 \right\rangle \quad \text{and} \quad \vec{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \right\rangle$$

Normal Plane: Use $\vec{T}(\pi/2)$:

$$\left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, 0 \right\rangle \cdot \left\langle x - \frac{\pi}{2}, y - 0, z - 3 \right\rangle = 0$$

Osculating Plane: Use $\vec{B}(\pi/2)$:

$$\left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0 \right\rangle \cdot \left\langle x - \frac{\pi}{2}, y - 0, z - 3 \right\rangle = 0$$

A comment on the normal and osculating planes:

Recall that we only need a vector which is perpendicular to the plane to find an equation for it, in particular, the length of the vector doesn't matter. So, easier vectors to use are:

Normal Plane: use $\vec{r}'(t)$

Osculating Plane: use $\vec{r}'(t) \times \vec{r}''(t)$

Extra Examples

1. Find an integral to evaluate the length of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ as t varies between 0 and 2π .
2. A particle has position $\mathbf{r}(t) = \langle t - \frac{2}{3}t^3, t^2, 2t \rangle$ at time t . Find an integral to evaluate the distance traveled by the particle as it moves from $(0, 0, 0)$ to $(\frac{1}{3}, 1, 2)$.
3. Let C be the curve given by the vector function $\mathbf{r}(t) = \langle 1 - \frac{1}{2}t^2, t, t^2 \rangle$. Find the equation of the osculating plane to C at the point $(1, 0, 0)$.
4. For the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ at the point $(1, \frac{2}{3}, 1)$. Also find equations of the normal and osculating planes at the same point.