University of Notre Dame Calculus III

LECTURE 8:

Arc Length and Curvature

In Calc II we found the arc length of a plane curve $x(t) = f(t)$, $y(t) = g(t)$, $a \le t \le b$ as

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{dx^2 + dy^2} \end{aligned}$$

This was done by approximating the curve by straight lines. We can do the same thing for curves in $\mathbb{R}^3$. This leads to:

Definition 1. *The* <u>*arc length*</u> *of the curve* $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$*,* $a \le t \le b$ *i*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

There are some technical assumptions to this formula:

- $\vec{r}(t)$' is nonzero.
- f', g', and h' must be continuous (i.e. $\vec{r}$ is C^1)

Notice that because

$$||\vec{r}'(t)|| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

we have that the arc length can be computed as

$$L = \int_a^b ||\vec{r}'(t)||\, dt$$

Example 1. Find the arc length of

$$\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$$

where $-5 \le t \le 5$.

Solution:

A curve C need not have a unique representation by a vector function, in fact, none do. For example $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \le t \le 2$ is also represented by $\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \le u \le 2$. $\vec{r}_1$ and $\vec{r}_2$ are called parametrizations of C. There is one particular parametrization we care about, and it is found as follows: Suppose C is given by $\vec{r}(t)$, $a \le t \le b$, with $\vec{r}\,'$ continuous and $\vec{r}(t)$ traverses C exactly once. We can define the arc length function

$$s(t) = \int_a^t ||\vec{r}\,'(u)||\,du$$

which tells us the distance traveled at time t. Now, suppose we can solve this equation for t in terms of s. In other words invert $s(t)$ so that $t = t(s)$. Then:

Definition 2. *The Arc Length Reparametrization of $\vec{r}(t)$ is*

$$\vec{r} = \vec{r}(t(s)), \quad 0 \le s \le L$$

where L = arc length of $\vec{r}$ from $t = a$ to $t = b$.

Example 2. Reparametrize $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$, $-5 \le t \le 5$ with respect to arc length.

Solution:

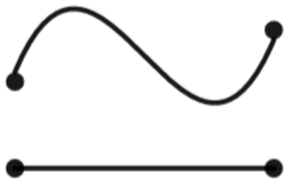
An interesting fact about arc length reparametrizations:

$$\frac{d\vec{r}}{s} = \frac{d}{ds}(\vec{r}(t(s))) = \left(\frac{dt}{ds}\right)\vec{r}\,'(t(s)) = \frac{1}{||\vec{r}\,'(t)||}\vec{r}\,'(t(s))$$

So, $\left|\left|\frac{d\vec{r}}{ds}\right|\right| = 1$, that is, arc length reparametrizations always move with unit speed!

Curvature

Intuitively, curvature is a measure of how sharply a curve bends. Pictorally, the curved line has larger curvature than the bottom line does:



Definition 3. *A parametrization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}'$ is continuous on I and $\vec{r}'(t) \neq \vec{0}$ for any $t \in I$. A curve C is called smooth if it has a smooth parametrization.*

We quantify curvature as the rate of change of the unit tangent vector with respect to arc length. In symbols, the curvature of $\vec{r}$ is

$$\boxed{\kappa = \left\| \frac{d\vec{T}}{ds} \right\|}$$

Now $\frac{d\vec{T}}{ds}$ can often be messy to compute, however, we have a trick: by the chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \frac{d\vec{T}}{ds}||\vec{r}'(t)||$$

So, a more convenient formula for curvature is

$$\boxed{\kappa = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||}}$$

Example 3. Find the curvature of a circle of radius a.

Solution:

Even that formula is more effort than needed. Another is

$$\boxed{\kappa(t) = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}}$$

Example 4. Find the curvature of $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ at $(0,1,1)$.

Solution:

In the special case of a plane curve $y = f(x)$, by parametrizing it as $\vec{r}(x) = \langle x, f(x) \rangle$ we get

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}\,.$$

Frenet-Serret Frame "$T-N-B$ Frame"

This consists of 3 vectors derived from a parametrization, $\vec{r}(t)$: $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$. We already know one of them, the other two are

Unit Normal Vector (Requiring $||\vec{T}'(t)|| \neq 0$, equivalently $\kappa(t) \neq 0$)

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

Binormal Vector

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Since $||\vec{T}(t)|| = 1$, we have $\vec{T}(t) \cdot \vec{T}'(t) = 0$, so $\vec{T} \perp \vec{N}$. By definition of ×, $\vec{B} \perp \vec{T}$, $\vec{N}$, so the three vectors are all orthogonal to each other. Thus since $||\vec{T}|| = ||\vec{N}|| = 1$, we have $||\vec{B}|| = ||\vec{T} \times \vec{N}|| = ||\vec{T}||\,||\vec{N}|| \sin\frac{\pi}{2} = 1$, thus all of $\vec{T}$, $\vec{N}$, and $\vec{B}$ and hard to compute, so here's an alternate way:

$$\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{||\vec{r}'(t) \times \vec{r}''(t)||} \qquad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t)\,.$$

$\vec{N}$ always points in the direction the curve is bending and $\vec{B}$ points orthoganal to the motion of the curve.

We can create some planes using $\vec{T}$, $\vec{N}$, and $\vec{B}$.

Normal Plane: This plane is perpendicular to $\vec{r}(t)$. It is determined by $\vec{N}$ and $\vec{B}$, and so has $\vec{T}$ as a vector orthogonal to it.

Osculating Plane: This plane best captures the motion of the curve. It is determined by $\vec{T}$ and $\vec{N}$, and so has $\vec{B}$ as a vector perpendicular to it.

Rectifying Plane: This plane determined by $\vec{T}$ and $\vec{B}$. We won't bother with this one.

Example 5. Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$ and find equations for the normal and osculating planes at $(\pi/2, 0, 3)$.

Solution:

A comment on the normal and osculating planes:
Recall that we only need a vector which is perpendicular to the plane to find an equation for it, in particular, the length of the vector doesn't matter. So, easier vectors to use are:
Normal Plane: use $\vec{r}'(t)$
Osculating Plane: use $\vec{r}'(t) \times \vec{r}''(t)$

Extra Examples

1. Find an integral to evaluate the length of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ as t varies between 0 and 2π.
2. A particle has position $\mathbf{r}(t) = \langle t - \frac{2}{3}t^3, t^2, 2t \rangle$ at time t. Find an integral to evaluate the distance traveled by the particle as it moves from $(0,0,0)$ to $\left(\frac{1}{3}, 1, 2\right)$.
3. Let C be the curve given by the vector function $\mathbf{r}(t) = \langle 1 - \frac{1}{2}t^2, t, t^2 \rangle$. Find the equation of the osculating plane to C at the point $(1,0,0)$.
4. For the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$. Find the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ at the point $\left(1, \frac{2}{3}, 1\right)$. Also find equations of the normal and osculating planes at the same point.