University of Notre Dame Calculus III

LECTURE 9: MOTION IN SPACE

Motion in Space

Suppose a particle moves along a trajectory $\vec{r}(t)$.
It's velocity is $\vec{v}(t) = \vec{r}\,'(t)$
acceleration is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$
and speed is $||\vec{v}(t)||$, which I wil denote by v.

Example 1. A particle has acceleration function

$$\vec{a}(t) = 4t\hat{i} + 6\sin t\,\hat{j} + e^t\hat{k}\ .$$

If its initial velocity is $\vec{v}(0) = 3\hat{j}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

Solution:

$$\vec{v}(t) = \int \vec{a}(t)\,dt = (2t^2 + C_1)\hat{i} + (-6\cos t + C_2)\hat{j} + (e^t + C_3)\hat{k}$$

$\vec{v}(0) = C_1\hat{i} + (-6 + C_2)\hat{j} + (1 + C_3)\hat{k} = 3\hat{j}$ meaning $C_1 = 0$, $C_2 = 9$, $C_3 = -1$ so

$$\vec{v}(t) = \int \vec{a}(t)\,dt = 2t^2\hat{i} + (9 - 6\cos t)\hat{j} + (e^t - 1)\hat{k}$$

Now

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{2}{3}t^3 + D_1\right)\hat{i} + (9t - 6\sin t + D_2)\hat{j} + (e^t - t + D_3)\hat{k}$$

$\vec{r}(0) = D_1\hat{i} + D_2\hat{j} + (1 + D_3)\hat{k} = \hat{0}$ meaning $D_1 = D_2 = 0$ and $D_3 = -1$ so

$$\vec{r}(t) = \frac{2}{3}t^3\hat{i} + (9t - 6\sin t)\hat{j} + (e^t - t - 1)\hat{k}\ .$$

If the particle has mass m and acceleration $\vec{a}(t)$, the force it experiences is given by Newton's second law:

$$\vec{F}(t) = m\vec{a}(t)\ .$$

Example 2. A projectile is fired with a muzzle speed 200 m/s and angle of elevation 60°. If the projectile is fired from a distance of $10m$ above ground level, what is the distance covered by the projectile? (All forces, except gravity, are assumed negligible.)

Solution:

The only force acting on the projectile is gravity, so $\vec{F}(t) = m\vec{a}(t) = -mg\hat{j}$. This means $\hat{a}(t) = -g\hat{j}$. So

$$\vec{v}(t) = \int \vec{a}(t) = -mg\hat{j} .$$

This means $\vec{a}(t) = -g\hat{j}$. So, $\vec{v}(t) = \int \vec{a}(t)\,dt = C_1\hat{i} + (-gt + C_2)\hat{j}$. To get the initial velocity, we use the given information $v = 200 = ||\vec{v}(0)||$. So

$$\vec{v}_0 = (200\cos 60°)\hat{i} + (200\sin 60°)\hat{j} = 100\hat{i} + 100\sqrt{3}\hat{j}$$

meaning $C_1 = 100$ and $C_2 = 100\sqrt{3}$. Thus the velocity function is

$$\vec{v}(t) = 100\hat{i} + (100\sqrt{3} - gt)\hat{j}$$

The position function is

$$\vec{r}(t) = \int \vec{v}(t)\,dt = (100t + D_1)\hat{i} + (100\sqrt{3}t - \frac{1}{2}gt^2 + D_2)\hat{j} .$$

The initial position is $\vec{r}(0) = 10\hat{j}$, so $D_1 = 0$, $D_2 = 10$. Thus

$$\vec{r}(t) = 100t\hat{i} + (100\sqrt{3}t - \frac{1}{2}gt^2 + 10)\hat{j} .$$

The particle hits the ground when the $\hat{j}$-component is 0: $100\sqrt{3}t - \frac{1}{2}gt^2 + 10 = 0$ meaning

$$t = \frac{-100\sqrt{3} \pm \sqrt{30000 + 20g}}{-g}$$

We take the positive value of t

$$t = \frac{-100\sqrt{3} - \sqrt{30000 + 20g}}{-g} \approx 35.4$$

Plugging this in the $\hat{i}$-component gives the distance traveled: dist $\approx 100(35.4)m \approx 3540m$.

Recall that the motion of a curve is best captured by the osculating plane at any point. (After all, $\vec{B}(t)$ is perpidicular to both $\vec{r}'(t)$ and $\vec{r}''(t)$.) We aim to write the acceleration in terms of $\vec{T}(t)$ and $\vec{N}(t)$. Let's start with $\vec{T}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{\vec{v(t)}}{v(t)}$$

So $\vec{v}(t) = v(t)\vec{T}(t)$.
Take a derivative:

$$\vec{v}''(t) = v'(t)\vec{T}(t) + v(t)\vec{T}'(t) = \vec{a}(t)$$

Now $\kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||} = \frac{||\vec{T}'(t)||}{v(t)}$ so $||\vec{T}'(t)|| = v(t)\kappa(t)$.
This allows us to write:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{\vec{T}'(t)}{v(t)\kappa(t)}$$

So

$$\vec{T}'(t) = v(t)\kappa(t)\vec{N}(t)\ .$$

Finally: $\vec{a}(t) = v'(t)\vec{T}(t) + (v(t))^2\kappa(t)\vec{N}(t)$.
This motivates the definitions: tangential component of acceleration $a_T = v'$ and normal component of acceleration $a_N = v^2\kappa$.
With a little work we see

$$a_T = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{||\vec{r}'(t)||} \qquad a_N = \frac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||}$$

A convenient fact:
$\vec{a} = a_T\vec{T} + a_N\vec{N}$ Since $\vec{T}\cdot\vec{N} = 0$ and $\vec{T}\cdot\vec{T} = \vec{N}\cdot\vec{N} = 1$.

$$||\vec{a}||^2 = \vec{a}\cdot\vec{a} = (a_T\vec{T} + a_N\vec{N})\cdot(a_T\vec{T} + a_N\vec{N}) = a_T^2 + a_N^2$$

So, $||\vec{a}|| = \sqrt{a_T^2 + a_N^2}$

Example 3. Find the normal and tangential components of acceleration for a particle moving along the trajectory $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$

Solution:

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$
$$\vec{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$$

$$\vec{r}'(t)\cdot\vec{r}''(t) = \sin t\cos t - \cos t\sin t + 0 = 0$$

So $a_T = 0$ meaning $||\vec{a}|| = \sqrt{a_T^2 + a_N^2} = \sqrt{a_N^2} = a_N$

$$a_N = ||\vec{a}(t)|| = ||\vec{r}''(t)|| = \sqrt{\cos^2 t + \sin^2 t + 0^2} = 1$$

Extra Examples

1. A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$ on the ground. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

2. For the curve $\mathbf{r}(t) = \langle 3\cos t, t^2 + 1, 3\sin t\rangle$ find the tangential component of acceleration at $t = 1$.

3. For the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$, find a_T and a_N at $t = 1$.

4. The acceleration of a particle is given by $\mathbf{a}(t) = \langle e^t, 0, \sin(t)\rangle$. Find the position function $\mathbf{r}(t)$ of the particle if at time $t = 0$ the particle was passing through the origin with the velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$