University of Notre Dame Calculus III

LECTURE 9: MOTION IN SPACE

Motion in Space

Suppose a particle moves along a trajectory $\vec{r}(t)$.
It's velocity is $\vec{v}(t) = \vec{r}\,'(t)$
acceleration is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$
and speed is $||\vec{v}(t)||$, which I wil denote by v.

Example 1. A particle has acceleration function

$$\vec{a}(t) = 4t\hat{i} + 6\sin t\hat{j} + e^t\hat{k}\ .$$

If its initial velocity is $\vec{v}(0) = 3\hat{j}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

Solution:

If the particle has mass m and acceleration $\vec{a}(t)$, the force it experiences is given by Newton's second law:

$$\vec{F}(t) = m\vec{a}(t)\ .$$

Example 2. A projectile is fired with a muzzle speed 200 m/s and angle of elevation 60°. If the projectile is fired from a distance of $10m$ above ground level, what is the distance covered by the projectile? (All forces, except gravity, are assumed negligible.)

Solution:

Recall that the motion of a curve is best captured by the osculating plane at any point. (After all, $\vec{B}(t)$ is perpidicular to both $\vec{r}'(t)$ and $\vec{r}''(t)$.) We aim to write the acceleration in terms of $\vec{T}(t)$ and $\vec{N}(t)$. Let's start with $\vec{T}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{\vec{v(t)}}{v(t)}$$

So $\vec{v}(t) = v(t)\vec{T}(t)$.
Take a derivative:

$$\vec{v}''(t) = v'(t)\vec{T}(t) + v(t)\vec{T}'(t) = \vec{a}(t)$$

Now $\kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||} = \frac{||\vec{T}'(t)||}{v(t)}$ so $||\vec{T}'(t)|| = v(t)\kappa(t)$.
This allows us to write:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{\vec{T}'(t)}{v(t)\kappa(t)}$$

So

$$\vec{T}'(t) = v(t)\kappa(t)\vec{N}(t) \; .$$

Finally: $\vec{a}(t) = v'(t)\vec{T}(t) + (v(t))^2\kappa(t)\vec{N}(t)$.
This motivates the definitions: tangential component of acceleration $a_T = v'$ and
normal component of acceleration $a_N = v^2\kappa$.
With a little work we see

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{||\vec{r}'(t)||} \qquad a_N = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||}$$

A convenient fact:
$\vec{a} = a_T \vec{T} + a_N \vec{N}$ Since $\vec{T} \cdot \vec{N} = 0$ and $\vec{T} \cdot \vec{T} = \vec{N} \cdot \vec{N} = 1$.

$$||\vec{a}||^2 = \vec{a} \cdot \vec{a} = (a_T \vec{T} + a_N \vec{N}) \cdot (a_T \vec{T} + a_N \vec{N}) = a_T^2 + a_N^2$$

So, $||\vec{a}|| = \sqrt{a_T^2 + a_N^2}$

Example 3. Find the normal and tangential components of acceleration for a particle moving along the trajectory $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Solution:

Extra Examples

1. A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$ on the ground. Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?
2. For the curve $\mathbf{r}(t) = \langle 3\cos t, t^2 + 1, 3\sin t \rangle$ find the tangential component of acceleration at $t = 1$.
3. For the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$, find a_T and a_N at $t = 1$.
4. The acceleration of a particle is given by $\mathbf{a}(t) = \langle e^t, 0, \sin(t) \rangle$. Find the position function $\mathbf{r}(t)$ of the particle if at time $t = 0$ the particle was passing through the origin with the velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$