University of Notre Dame Calculus III

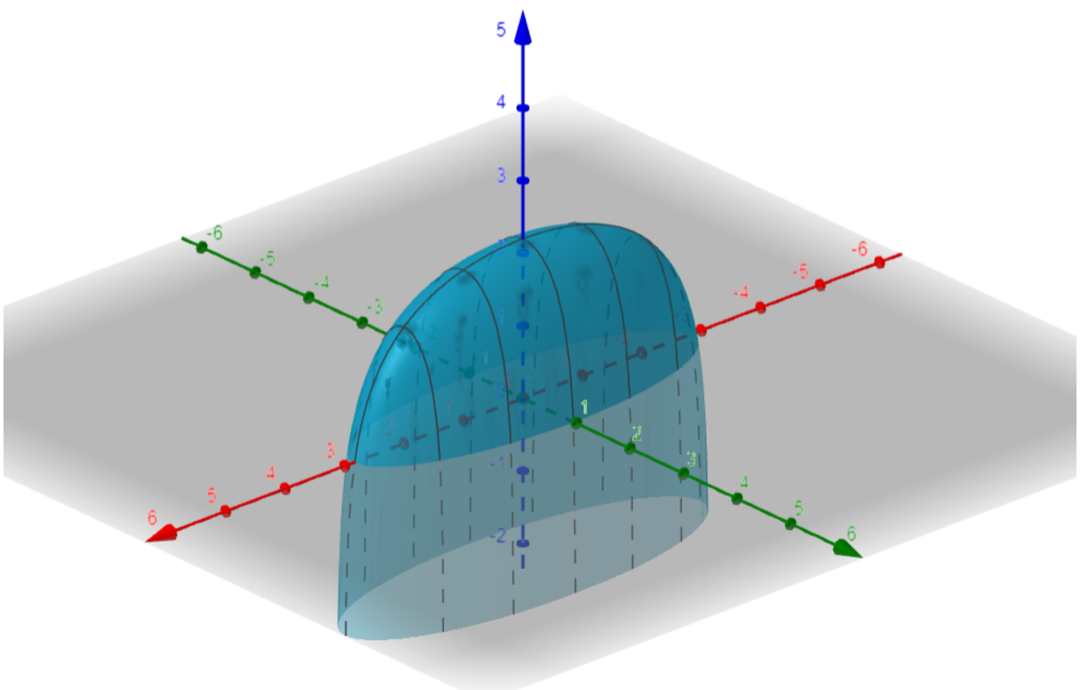
Functions of Several Variables

A function of two variables, f, is a rule which assigns to each pair (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number $f(x, y)$. D is called the domain of f and the range of f is the set of values it takes on, i.e., $\{f(x, y)|(x, y) \in D\}$.
If we write $z = f(x, y)$, x and y are the independent variables and z is the dependent variable.

Example 1. Find the domain of $f(x, y) = \ln(9 - x^2 - 9y^2)$ and sketch it.

Solution:

Recall that $\ln(t)$ is defined for $t > 0$. So, we need $9 - x^2 - 9y^2 > 0$ iff $x^2 + 9y^2 < 9$ iff $\frac{x^2}{9} + y^2 < 1$. The domain is thus $D = \{(x, y)|\frac{x^2}{9} + y^2 < 1\}$ and a sketch is:



Example 2. Sketch the graph of $h(x, y) = 4x^2 + y^2$.

Solution:

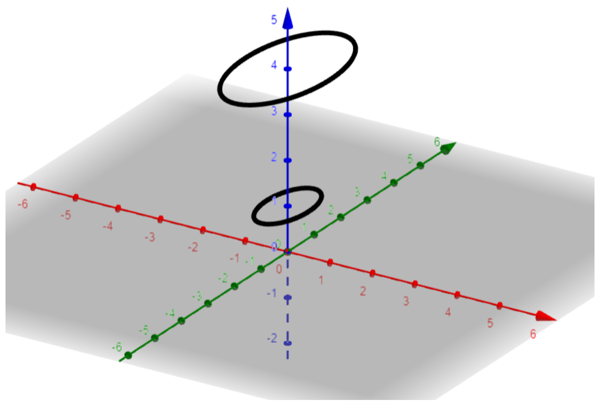
The graph is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = h(x, y)$. We want to graph $z = 4x^2 + y^2$. We will see what this looks like using level curves, that is, taking slices of the graph at different z-values. Note that $z \geq 0$ here.

$z = 0$: $0 = 4x^2 + y^2$ implying $x = y = 0$. So the level curve for $z = 0$ is the origin $(0, 0)$

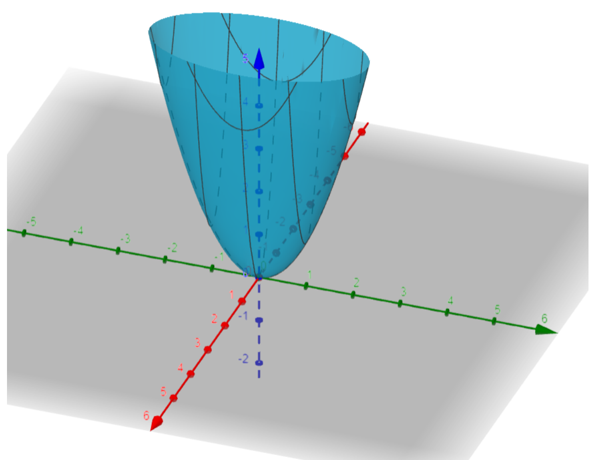
$z = 1$: $1 = 4x^2 + y^2$ iff $\frac{x^2}{(\frac{1}{2})^2} + y^2 = 1$ So the level curve at height $z = 1$ is an ellipse.

$z = 4$: $4 = 4x^2 + y^2$ iff $x^2 + \frac{y^2}{4} = 1$ An ellipse again.

Putting these together gives



The way to read this graph is to realize that the level curve you sit on determines your height. Lifting the level curves to their respective z-values, we see the elliptic paraboloid we expect:



The 2D picture is called the contour map of h.

The level curves are sometimes called contours. You would see this kind of plot when you look at temperature maps or topography maps.

Functions of Three or More Variables

A function, f, of three variables will take in a triple (x, y, z) and output a real number $f(x, y, z)$. The set of allowable inputs points (x, y, z) is its domain. Notice that we cannot graph a function of 3 variables since its graph is 4 dimensional!
The best we can do is study "pictures of it in time", i.e. sets $f(x, y, z) = c$. In this case they are called level surfaces.
For functions of more than 3 variables, sets of points $(x_1, \cdots, x_n, c)$ are called level sets of the function. They're going to be impossible to plot.

Limits and Continuity

Definition 1. *Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approches (a, b) is L if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$. If this is so we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L\,.$$

Intuitively, this means as the inputs of f get closer to (a, b), the values f can take on are "squeezed towards L". These limits are exceptionally hard to compute, in general. This is because, in $\mathbb{R}^2$, there are more than just two paths to (a, b), there are infinitely many!
Based on this, it's easier usually to show limits don't exist. This is because showing that the limit along two different paths are not equal is easier than showing all of the limits along all of the infinitely many paths are the same.

Example 3. Does the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist?

Solution:

Let travel along the positive x-axis: this means we use the points $(x, 0)$, $x > 0$ and let $x \to 0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

now along the positive y-axis:

$$\lim_{(0,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$1 \neq -1$, so the limit does not exist!

Be careful though, this x- and y-axis trick doesn't always work. For example:

Example 4. Does $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Solution:

If we plug in the x- or y-axis, the top becomes zero, so the limits along the axes are zero. However, along $y = x$:

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

So, the limit does not exist.

A convenient trick to use to show limits do not exist is to check along the line $y = mx$, where m is arbitrary (This is for limits to $(0,0)$.) If there is a dependence on m, the limit does not exist. Consider again the last example. Along $y = mx$ we have

$$\lim_{(x,mx)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

since m is arbitrary, this limit does not exist.
This trick, however, is not cure-all.

Example 5. Does $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Solution:

Along $y = mx$:

$$\lim_{(x,mx)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0\ .$$

But, along $x = y^2$,

$$\lim_{(y^2,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2} \neq 0$$

So the limit does not exist.

Definition 2. *A function $f(x,y)$ is continuous at (a,b) if $\lim_{(a,b)\to(a,b)} f(x,y) = f(a,b)$. We say f is continuous on D if it is continuous at every point in D.*

Facts for Checking for Continuity:

- Polynomials are continuous
- Sums of continuous functions are continuous
- Products of continuous functions functions are continuous
- Quotients of continuous functions are continuous when the denominator is $\neq 0$.
- The composition of continuous functions is continuous

Example 6. Does $\lim_{(x,y)\to(0,0)} \frac{4-xy}{x^2+3y^2+1}$ exist?

Solution:

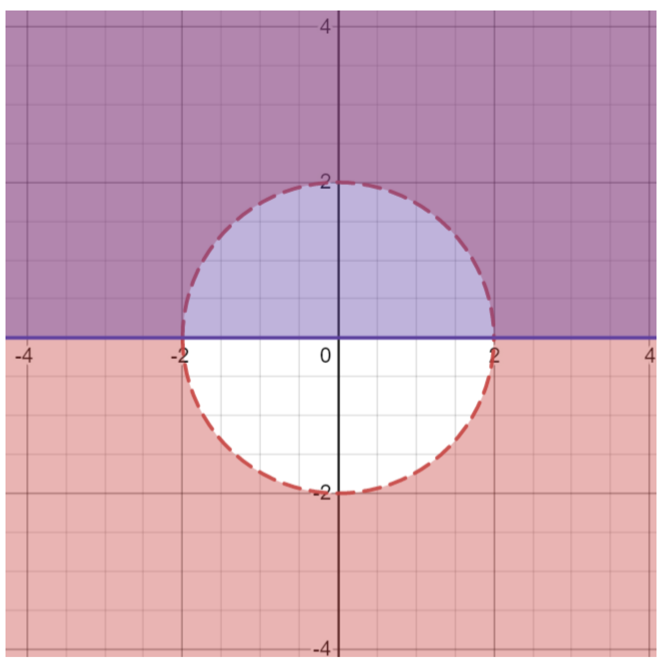
$f(x,y) = \frac{4-xy}{x^2+3y^2+1}$ is continuous at $(0,0)$ since it is a quotient of polynomials and the denominator isn't zero at $(0,0)$. Thus the limit exists and its value is $f(0,0) = \frac{4-0}{0+0+1} = 4$.

Example 7. Determine where $g(x,y) = \ln(x^2+y^2-4) - \sqrt{y}$ is continuous

Solution:

1. $\ln(t)$ is continuous for $t > 0$
2. $\ln(x^2+y^2-4)$ is continuous for $x^2+y^2-4 > 0$
3. $\sqrt{y}$ is continuous for $y \geq 0$

So, g is continuous where $x^2+y^2-4 > 0$ (the red part of the graph below) and $y \geq 0$ (the purple part of the graph below). Graphically:



g is continuous on the intersection* of the two regions.

Extra Problems

1. Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < x < y^2\}$. Show every line through $(0,0)$ contains an interval around $(0,0)$ that is in $\mathbb{R}^2 - A$.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = 0$ if $x \notin A$, and $f(x) = 1$ if $x \in A$. For $\vec{h} \in \mathbb{R}^2$, define $g_{\vec{h}}(t)$ by $g_{\vec{h}}(t) = f(t\vec{h})$. Show for all $\vec{h} \in \mathbb{R}^2$ that $g_{\vec{h}}(t)$ is continuous at 0, but that $f(x)$ is not continuous at $(0,0)$.