University of Notre Dame Calculus III

Partial Derivatives

Definition 1. *The partial derivative of $f = f(x, y)$*

- *with respect to x is*
$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
- *with respect to y is*
$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Naturally, these are not the most practical way to compute partial derivatives. Notice that, for example in $\frac{\partial f}{\partial x}$, the limit has no concern for y. This means we can compute $\frac{\partial f}{\partial x}$ by pretending y is just a constant and taking the derivative with respect to x. Likewise for finding $\frac{\partial f}{\partial y}$.

Ex: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = x^3 + 2x^2 y + x^4 y^2 + \sqrt{y}$

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy + 4x^3 y^2 \qquad \frac{\partial f}{\partial y} = 2x^2 + 2x^4 y + \frac{1}{2}\frac{1}{\sqrt{y}}$$

What are the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to be interpreted as? They are the rates of change of f in the x- an y-directions, respectively. Visually, we see this number as the slope of the curve on the surface, passing through the point in either the x- or y-direction.

Example 1. Find the first partials of $f(x, y) = \sin(x \cos y)$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos(x \cos y) \cdot \left(\frac{\partial}{\partial x}(x \cos y)\right) \\
&= \cos(x \cos y) \cos y \\
\frac{\partial f}{\partial y} &= \cos(x \cos y) \cdot \left(\frac{\partial}{\partial y}(x \cos y)\right) \\
&= -\cos(x \cos y)(x \sin y)
\end{aligned}$$

We can also do implicit differentiation.

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where z is implicitly defined by $yz + x\ln y = z^2$.

Solution:

Take the derivative of both sides w.r.t. x:

$$y\frac{\partial z}{\partial x} + \ln(y) = 2z\frac{\partial z}{\partial x}$$

Solve for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}\ .$$

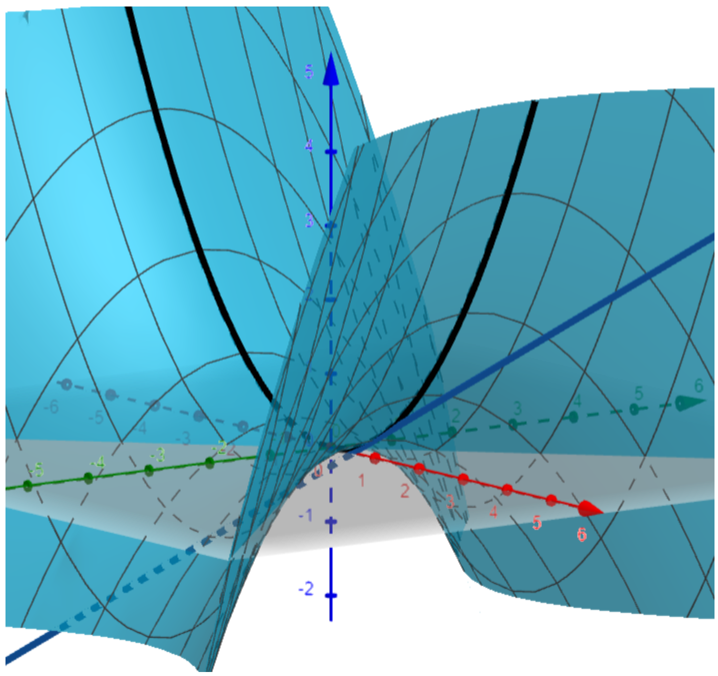
For $\frac{\partial z}{\partial y}$ differentiate both sides w.r.t. y:

$$\left(z + y\frac{z}{y}\right) + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

and solve for $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = \frac{z + \left(\frac{x}{y}\right)}{2z - y} = \frac{yz + x}{2yz - y^2}$$

We can, of course, take partial derivatives of functions of 3 or more variables. The procedure is exactly the same: treat all the variables as constants except the one you're taking the derivative with respect to, then differentiate.



Above is the graph of the function $f(x,y) = -\frac{1}{4}(x^2 - y^2)$, with the tangent line of the function when $y = 0$ (in blue). The slope of this line is the partial derivative of f with respect to x evaluated at $(x,y) = (1,0)$.

Higher-Order Derivatives

Naturally, we can take derivatives of derivatives. The notation is this

$$\left(f_x\right)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$\left(f_x\right)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Notice the change of order between the subscript notation and Leibniz notation. With this notation we can take higher partials of any finite degree.

Example 3. Find all second partials of $f(x, y) = x^4y^3 - y^4$

Solution:

$$\begin{aligned} f_x &= 4x^3y^3 & f_y &= 3x^4y^2 - 4y^3 \\ f_{xx} &= 12x^2y^3 & f_{yy} &= 6x^4y - 12y^2 \\ f_{xy} &= 12x^3y^2 & f_{yx} &= 12x^3y^2 \end{aligned}$$

Notice that $f_{xy} = f_{yx}$. This is not a coincidence

Theorem 2. *Clairaut's Theorem: Suppose f is defined on a disk D containing the point (a, b). If f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}$$

Chain Rule

Theorem 3. *(Proper) Chain Rule: Suppose $\vec{F}(x_1, ..., x_n) = \langle f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)\rangle$ and $\vec{G}(y_1, ..., y_p) = \langle g_1(y_1, ..., y_p), ..., g_n(y_1, ..., y_p)\rangle$ and that the range of $\vec{G}$ is inside the domain of $\vec{F}$. If $\vec{G}$ is differentiable at $(a_1, ..., a_p) = \vec{a}$ and $\vec{F}$ is differentiable at $\vec{G}(\vec{a})$, then*

$$[D(\vec{F} \circ \vec{G})](\vec{a}) = D\vec{F}(\vec{G}(\vec{a})) \cdot D\vec{G}(\vec{a})$$

Let's reduce this now to the case when the outside function is scalar valued:

Theorem 4. *Chain Rule:*

$$\frac{\partial z}{\partial y_i}(\vec{a}) = \nabla f(\vec{G}(\vec{a})) \cdot \frac{\partial \vec{G}}{\partial y_i}(\vec{a})$$

writing this out, we find:

$$\frac{\partial z}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i}$$

Let's bring this back down to earth with an example

Example 4. Let $z = f(x, y) = x^2 + y^2 + xy$ and suppose $x = \sin t$, $y = e^t$. Find $\frac{dz}{dt}$.

Solution:

$$\frac{dz}{dt} = 2\sin t \cos t + e^t \cos t + 2e^{2t} + e^t \sin t$$

Example 5. Let $z = f(x, y) = x^2 - 2xy + y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Find z_r and r_θ.

Solution:

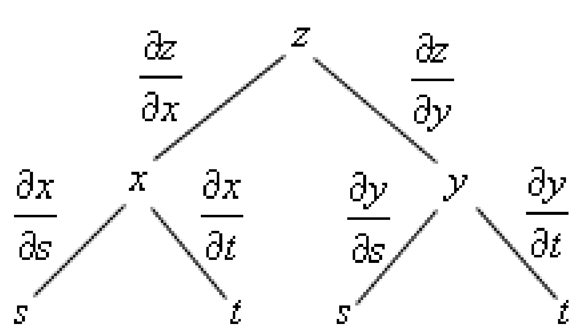
It's sometimes more convenient to leave f and its derivatives in terms of x and y and plug them in at the end.

$$f_x = 2x - 2y \text{ and } f_y = 2y - 2x \qquad \vec{G}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$$
$$\vec{G}_r = \langle\cos\theta, \sin\theta\rangle \qquad \vec{G}_\theta = \langle -r\sin\theta, r\cos\theta\rangle$$

$$\begin{aligned}\frac{\partial z}{\partial r} &=< f_x, f_y > \cdot \vec{G}_r = (2x-2y)\cos\theta + (2y-2x)\sin\theta \\ &= (2x-2y)(\cos\theta - \sin\theta) = (2r\cos\theta - 2r\sin\theta)(\cos\theta - \sin\theta) \\ &= 2r(\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta) = 2r(1-\sin 2\theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial \theta} &=< f_x, f_y > \cdot \vec{G}_\theta = (2x-2y)(-r\sin\theta) + (2y-2x)(r\cos\theta) \\ &= 2r(y-x)(\sin\theta + \cos\theta) = 2r(r\sin\theta - r\cos\theta)(\sin\theta + \cos\theta) \\ &= 2r^2(\sin\theta - \cos\theta)(\sin\theta + \cos\theta) = 2r^2(\sin^2\theta - \cos^2\theta) \\ &= -2r^2\cos 2\theta\end{aligned}$$

There is a useful bookkeeping method we can use for finding derivatives using the chain rule. This is using dependency trees. As in the last example:



Then, to find, say $\frac{\partial z}{\partial r}$, we follow the paths from z to r, each edge corresponding to a derivative to be taken, then adding up the paths. In this case

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Example 6. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where $z = \tan(uv)$ and $u = 2s + 3t$, and $v = 3s - 2t$

Solution:

We have:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = (v\sec^2(uv))(2) + (u\sec^2(uv))(3)\\ &= \sec^2(uv)(2v + 3u) = \sec^2((2s+3t)(3s-2t))(6s - 4t + 6s + 9t)\\ &= (12s + 5t)\sec^2(6s^2 + 5st - 6t^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = (v\sec^2(uv))(3) + (u\sec^2(uv))(-2)\\ &= (3v - 2u)\sec^2(uv) = (9s - 6t - 4s - 6t)\sec^2((2s+3t)(3s-2t))\\ &= (5s - 12t)\sec^2(6s^2 + 5st - 6t^2)\end{aligned}$$

Implicit Differentiation

The chain rule can make implicit differentiation. Let's suppose z is defined implicitly by $F(x, y, z) = 0$. Using the chain rule:

Taking the derivative with respect to x we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

meaning

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Likewise, we can find $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Ex: Find all first partials of z where z is defined implicitly by $x^3 z + y\cos z + \frac{\sin y}{z} = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 z}{x^3 - y\sin z - \frac{\sin y}{z^2}} = \frac{-3x^2 z^3}{x^3 z^2 - yz^2 \sin z - \sin y}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos z + \frac{\cos y}{z}}{x^3 - y\sin z - \frac{\sin y}{z^2}} = \frac{-(z^2 \cos z + z\cos y)}{x^3 z^2 - yz^2 \sin z - \sin y}$$

There are technical assumptions one needs for implicit differentiation. These are encapsulated in the implicit function theorem.

Example 7. Suppose we have a box containing $42in^3$ of an incompressible fluid. Holding the height of the box fixed at $7in$, we squeeze the width of the box so that it decreases at a rate of $1in/min$. How fast is the length of the box changing when the width in $3in$?

Solution:

Volume: $V = lwh$. We want $\frac{\partial l}{\partial t}$. l, w, and h depend on t. Take $\frac{d}{dt}$ of both sides:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ &= wh\frac{dl}{dt} + lh\frac{dw}{dt} + lw\frac{dh}{dt}\end{aligned}$$

Fluid incompressible means $\frac{dV}{dt} = 0$, height fixed means $\frac{dh}{dt} = 0$. We have $\frac{dw}{dt} = -1$. $h = 7$, when $w = 3$, $l = 2$ since $V = lwh = 42$. So

$$0 = (3)(7)\frac{dl}{dt} + (2)(7)(-1) + (2)(3)(0) = 21\frac{dl}{dt} - 14$$

So

$$\frac{dl}{dt} = \frac{14}{21}in/min = \frac{2}{3}in/min$$

Extra Problems

1. Let $f : \mathbb{R}^n \to \mathbb{R}$. Show that $\lim_{h\to 0} \frac{f(x_1+h_1, x_2+h_2+\ldots+x_n+h_n) - f(x_1, x_2, \ldots, x_n)}{h} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \ldots + \frac{\partial f}{\partial x_n}$.

2. Find expressions for the partial derivatives of the following functions:

   $F(x, y) = f(g(x)k(y), g(x) + h(y))$

   $F(x, y, z) = (x^y, y^z, z^x)$

   $F(x, y) = f(x, g(x), h(x, y))$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$, the limit $\lim_{t\to 0} \frac{f(a+tx)-f(a)}{t}$, if it exists, is denoted as $D_x f(a)$, and is called the directional derivative of f at a, in direction x. Show:

   $D_y f(a) = \frac{\partial f}{\partial y}(a)$, where y is the unit vector $(0, 1, 0, \ldots, 0)$

   $D_{ty} f(a) = t D_y f(a)$, if $t \in \mathbb{R}$

   If f is differentiable at a, show $D_{x+y} f(a) = D_x f(a) + D_y f(a)$