University of Notre Dame Calculus III

LECTURE 12: DIRECTIONAL DERIVATIVES

Directional Derivatives

The partial derivatives give slopes in the x- and y-direction but what about all the other directions? Pick a direction by choosing a unit vector $\vec{u} = \langle a, b\rangle$, then the directional derivative of $f = f(x, y)$ in the direction $\vec{u}$ is

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = f_x(x,y)a + f_y(x,y)b}$$

Writing $\vec{x} = \langle x, y\rangle$, we can write $D_{\vec{u}} f(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$. Naturally, this extends to a function of more variables.
Again, $\vec{u}$ must be a unit vector!

We can write the directional derivative in a more efficient way by defining The Gradient.

Gradients

If $f = f(x, y)$, the gradient of f is

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$$

if $f = f(x, y, z)$

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$$

Example 1. Find the gradient of $f(x, y, z) = xe^{xyz}$.

Solution:

$$\boxed{\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle = \langle e^{xyz} + xyze^{xyz}, x^2ze^{xyz}, x^2ye^{xyz}\rangle}$$

Thus we can rewrite the directional derivative in the form

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

where $\vec{u}$ is a unit vector.

Example 2. Find the directional derivative of $f(x,y) = e^x \sin y$ at $(3, \frac{\pi}{6})$ in the direction of $\vec{v} = \langle 1, -2 \rangle$.

Solution:

$\nabla f = \langle e^x \sin y, e^x \cos y \rangle$ so $\nabla f(3, \frac{\pi}{6}) = \langle e^3 \cdot \frac{1}{2}, e^3 \cdot \frac{\sqrt{3}}{2} \rangle = \frac{e^3}{2} \langle 1, \sqrt{3} \rangle$.
$\vec{v}$ is not a unit vector, so we have to make it one

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 1, -2 \rangle}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$$

Then

$$D_{\hat{v}} f\left(3, \frac{\pi}{6}\right) = \nabla f\left(3, \frac{\pi}{6}\right) \cdot \hat{v} = \frac{e^3}{2\sqrt{5}} [(1)(1) + (\sqrt{3})(-2)] = \frac{e^3}{2\sqrt{5}} (1 - 2\sqrt{3})$$

We can ask: in what direction if f changing the fastest? Let $\vec{u}$ be a unit vector and θ the angle between ∇f and $\vec{u}$. Then $D_{\vec{u}} f = \nabla f \cdot \vec{u} = ||\nabla f|| \, ||\vec{u}|| \cos\theta = ||\nabla f|| \cos\theta$. This is maximized when $\theta = 0$ ($\cos\theta = 1$), meaning that $\vec{u}$ points in the same direction as ∇f. Moreover, we see that the maximum rate of change is $||\nabla f||$.

Example 3. Find the maximum rate of change of $f(x,y,z) = \frac{x+y}{z}$ at $(1,1,-1)$, and the direction in which it occurs.

Solution:

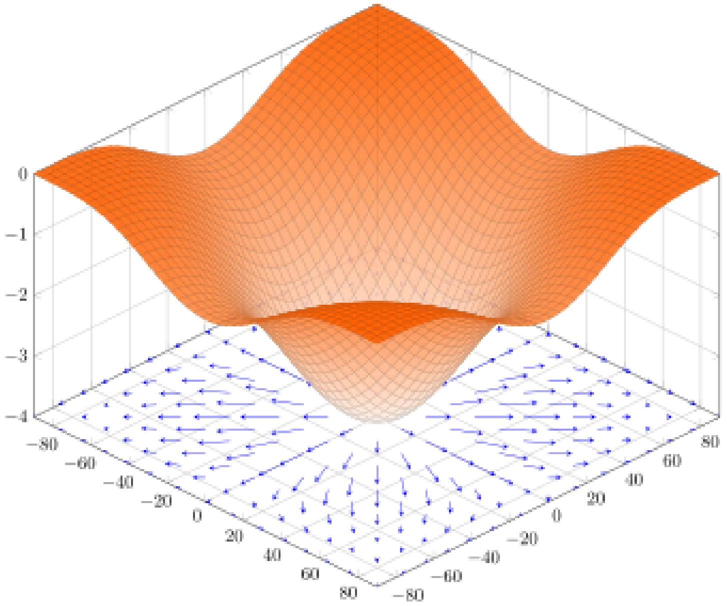
$$\nabla f = \langle \frac{1}{z}, \frac{1}{z}, \frac{-(x+y)}{z^2} \rangle$$

$\nabla f(1,1,-1) = \langle -1,-1,-2 \rangle$. The maximum rate of change is:

$$||\nabla f(1,1,-1)|| = ||\langle -1,-1,-2 \rangle = \sqrt{1+1+4} = \sqrt{6}$$

And the direction is:

$$\vec{u} = \frac{\nabla f(1,1,-1)}{||\nabla f(1,1,-1)||} = -\frac{1}{\sqrt{6}}\langle 1,1,2 \rangle$$



Above is the graph of the function $f(x,y) = -(\cos(x)^2 + \cos(y)^2)$, and below the graph, projected on the xy-plane, is the gradient field of the function.

Some Problems

1. Calculate ∇f if $f(x,y,z,w) = x^y + y^z - w^2 + z + 1$
2. Let $f(x,y,z) = (xz, y, zx^2)$ and $g(x,y,z) = x+y+z+3$. Find $\nabla f \cdot \nabla g$.
3. Prove the equality $\lim_{h\to 0} \frac{f(x+ah,y+bh)-f(x,y)}{h} = \nabla f(x,y)\cdot \vec{u}$, where $\vec{u} = (a,b)$.

   HINT: Use problem 1 from the last lecture!