

# University of Notre Dame Calculus III

## LECTURE 12: DIRECTIONAL DERIVATIVES

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### Directional Derivatives

The partial derivatives give slopes in the  $x$ - and  $y$ -direction but what about all the other directions? Pick a direction by choosing a unit vector  $\vec{u} = \langle a, b \rangle$ , then the directional derivative of  $f = f(x, y)$  in the direction  $\vec{u}$  is

$$D_{\vec{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x + ah, y + bh) - f(x, y)}{h} = f_x(x, y)a + f_y(x, y)b$$

Writing  $\vec{x} = \langle x, y \rangle$ , we can write  $D_{\vec{u}}f(\vec{x}) = \lim_{h \rightarrow 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$ . Naturally, this extends to a function of more variables.

Again,  $\vec{u}$  must be a unit vector!

We can write the directional derivative in a more efficient way by defining The Gradient.

### Gradients

If  $f = f(x, y)$ , the gradient of  $f$  is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

if  $f = f(x, y, z)$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

**Example 1.** Find the gradient of  $f(x, y, z) = xe^{xyz}$ .

**Solution:**

Thus we can rewrite the directional derivative in the form

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

where  $\vec{u}$  is a unit vector.

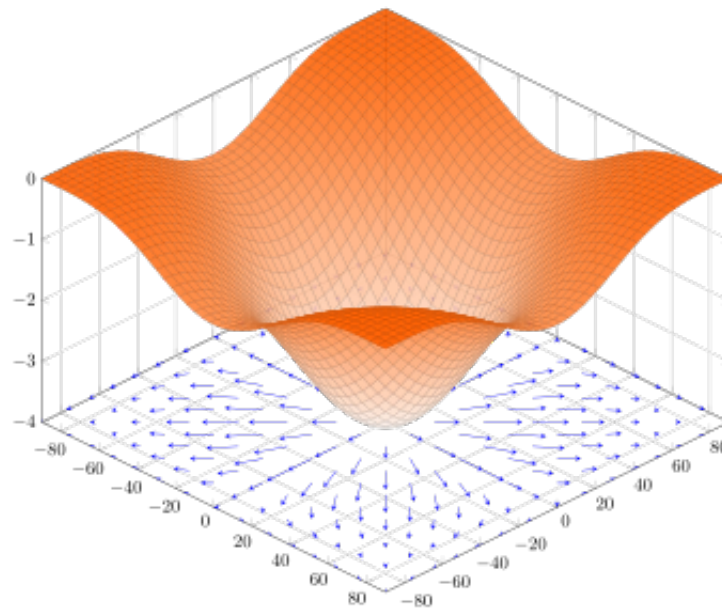
**Example 2.** Find the directional derivative of  $f(x, y) = e^x \sin y$  at  $(3, \frac{\pi}{6})$  in the direction of  $\vec{v} = \langle 1, -2 \rangle$ .

**Solution:**

We can ask: in what direction is  $f$  changing the fastest? Let  $\vec{u}$  be a unit vector and  $\theta$  the angle between  $\nabla f$  and  $\vec{u}$ . Then  $D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$ . This is maximized when  $\theta = 0$  ( $\cos\theta = 1$ ), meaning that  $\vec{u}$  points in the same direction as  $\nabla f$ . Moreover, we see that the maximum rate of change is  $\|\nabla f\|$ .

**Example 3.** Find the maximum rate of change of  $f(x, y, z) = \frac{x+y}{z}$  at  $(1, 1, -1)$ , and the direction in which it occurs.

**Solution:**



Above is the graph of the function  $f(x, y) = -(\cos(x)^2 + \cos(y)^2)$ , and below the graph, projected on the xy-plane, is the gradient field of the function.

### Some Problems

1. Calculate  $\nabla f$  if  $f(x, y, z, w) = x^y + y^z - w^2 + z + 1$
2. Let  $f(x, y, z) = (xz, y, zx^2)$  and  $g(x, y, z) = x + y + z + 3$ . Find  $\nabla f \cdot \nabla g$ .
3. Prove the equality  $\lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h} = \nabla f(x, y) \cdot \vec{u}$ , where  $\vec{u} = (a, b)$ .

HINT: Use problem 1 from the last lecture!