University of Notre Dame Calculus III

LECTURE 13:APPLICATIONS FOR THE GRADIENT

Geometric Uses of the Gradient

Suppose we have a surface S given as a levle surface of some function $F(x,y,z)$. So S is the graph of $F(x,y,z)=k$. Let's take a point (x_0,y_0,z_0) on S. To get a tangent vector to S at (x_0,y_0,z_0), take a curve, $\vec{r}(t)$ on S passing through the point (i.e. $F(\vec{r}(t))=k$ and $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$), then take its derivative at that point. That is the tangent vector is $\vec{r}'(t_0)$. If $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$, then by taking $\frac{d}{dt}$ of $F(\vec{r}(t))=k$ gives

$$\frac{dF}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}=0$$

So $\nabla F\cdot\vec{r}'(t)=0$, i.e., ∇F is perpendicular to every tangent vector on S, i.e., ∇F is perpendicular to S. This, of course, applies to show ∇G is perpendicular to level curves of $G(x,y)$.

Example 1. Find the tangent plane and normal line to the surface $y=x^2-z^2$ at $(4,7,3)$.

Solution:

First, notice $y=x^2-z^2$ is a level surface of $F(x,y,z)=x^2-y-z^2$, namely $F(x,y,z)=0$.

$$\nabla F=2x,-\vec{1},-2z \qquad\qquad \nabla F(4,7,3)=\langle 8,-1,-6\rangle\ .$$

The tangent plane has normal vector $\nabla F(4,7,3)$ and contains $(4,7,3)$, thus an equation is

$$\langle 8,-1,-6\rangle\cdot\langle x-4,y-7,z-3\rangle=0$$

or

$$8x-y-6z=7$$

The normal line points in the direction of $\nabla F(4,7,3)$ and contains $(4,7,3)$, thus an equation is

$$\vec{l}(t)=\langle 4,7,3\rangle+t\langle 8,-1,-6\rangle=\langle 4+8t,7-t,3-6t\rangle$$

Another use:

Example 2. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ is tangent to the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ at $(1, 1, 2)$.

Solution:

The ellipsoid is level surface of $F(x, y, z) = 3x^2 + 2y^2 + z^2$ and the sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z$ Now

$$\nabla F = \langle 6x, 4y, 2z\rangle \qquad \nabla F(1,1,2) = \langle 6, 4, 4\rangle$$
$$\nabla G = \langle 2x - 8, 2y - 6, 2z - 8\rangle \qquad \nabla G(1,1,2) = \langle -6, -4, -4\rangle$$

Since $\nabla F(1, 1, 2) = -\nabla G(1, 1, 2)$, the surfaces are tangent at $(1, 1, 2)$.

One more use:

Example 3. Find an equation for the tangent line to the intersection* of the hyperbolid $x^2 - y^2 + z^2 = 6$ and the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, 2, 3)$.

Solution:

The hyperboloid is a level surface of $F(x, y, z) = x^2 - y^2 + z^2$
The sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2$
A normal vector to the hyperboloid at $(1, 2, 3)$ is $\nabla F(1, 2, 3) = \langle 2, -4, 6\rangle$
A normal vector to the sphere at $(1, 2, 3)$ is $\nabla G(1, 2, 3) = \langle 2, 4, 6\rangle$
Since $\nabla F(1, 2, 3)$ and $\nabla G(1, 2, 3)$ are both perpendicular to the curve of intersection*, a vector tangent to it is

$$\nabla F(1,2,3) \times \nabla G(1,2,3) = \langle -48, 0, 16\rangle$$

So, the tangent line is represented by:

$$\vec{l}(t) = \langle 1, 2, 3\rangle + t\langle -48, 0, 16\rangle = \langle 1 - 48t, 2, 3 + 16t\rangle$$

Extra Example Problems

1. Find the directional derivative $D_{\mathbf{u}}f$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ makes angle $\pi/6$ with the positive x-axis.

2. A function $f(x,y,z)$ has gradient vector $(1,5,-2)$ at a point p. What is the directional derivative of f at p, in the direction of $(1,1,0)$?

3. Find the direction of maximal increase for the function $f(x,y) = xe^y$ at the point $(2,0)$.

4. A fly is in a room with temperature distribution (in celsius) given by

$$T(x,y,z) = \frac{1}{1+x^2+2y^2+3z^2}.$$

If the fly is at the point $(1,1,1)$, what direction should the fly move to decrease temperature the fastest? What is the rate of temperature decrease (in celsiuis/second) if the fly moves in this direction with speed 2 meters/second?

5. A state park has a topographical height map given by $h(x,y) = x^3 + 4y^2 - 2y$, where x, y, h are all in meters. How steep (in the sense of meters of height gained per horiztonal meter) is the mountainside at the point $(1,-1)$? There is a path starting at $(0,0)$ that has constant elevation. What is the initial direction of this path?

6. Find the tangent plane to the level surface $y - x^2 - 5z^2 = 1$ at the point $(1,6,1)$.

7. At what point(s) on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

8. Find vector equation of tangent line at $(1,1,-3)$ to curve defined by the intersection* of surfaces $x^2 + 2y^2 = 3$, $x + 2y + z = 0$.