University of Notre Dame Calculus III

LECTURE 13:APPLICATIONS FOR THE GRADIENT

Geometric Uses of the Gradient

Suppose we have a surface S given as a levle surface of some function $F(x, y, z)$. So S is the graph of $F(x, y, z) = k$. Let's take a point (x_0, y_0, z_0) on S. To get a tangent vector to S at (x_0, y_0, z_0), take a curve, $\vec{r}(t)$ on S passing through the point (i.e. $F(\vec{r}(t)) = k$ and $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$), then take its derivative at that point. That is the tangent vector is $\vec{r}'(t_0)$. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then by taking $\frac{d}{dt}$ of $F(\vec{r}(t)) = k$ gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

So $\nabla F \cdot \vec{r}'(t) = 0$, i.e., ∇F is perpendicular to every tangent vector on S, i.e., ∇F is perpendicular to S. This, of course, applies to show ∇G is perpendicular to level curves of $G(x, y)$.

Example 1. Find the tangent plane and normal line to the surface $y = x^2 - z^2$ at $(4, 7, 3)$.

Solution:

Another use:

Example 2. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ is tangent to the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ at $(1, 1, 2)$.

Solution:

One more use:

Example 3. Find an equation for the tangent line to the intersection* of the hyperbolid $x^2 - y^2 + z^2 = 6$ and the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, 2, 3)$.

Solution:

Extra Example Problems

1. Find the directional derivative $D_{\mathbf{u}}f$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ makes angle $\pi/6$ with the positive x-axis.

2. A function $f(x,y,z)$ has gradient vector $(1,5,-2)$ at a point p. What is the directional derivative of f at p, in the direction of $(1,1,0)$?

3. Find the direction of maximal increase for the function $f(x,y) = xe^y$ at the point $(2,0)$.

4. A fly is in a room with temperature distribution (in celsius) given by

$$T(x,y,z) = \frac{1}{1+x^2+2y^2+3z^2}.$$

   If the fly is at the point $(1,1,1)$, what direction should the fly move to decrease temperature the fastest? What is the rate of temperature decrease (in celsiuis/second) if the fly moves in this direction with speed 2 meters/second?

5. A state park has a topographical height map given by $h(x,y) = x^3 + 4y^2 - 2y$, where x, y, h are all in meters. How steep (in the sense of meters of height gained per horiztonal meter) is the mountainside at the point $(1,-1)$? There is a path starting at $(0,0)$ that has constant elevation. What is the initial direction of this path?

6. Find the tangent plane to the level surface $y - x^2 - 5z^2 = 1$ at the point $(1,6,1)$.

7. At what point(s) on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

8. Find vector equation of tangent line at $(1,1,-3)$ to curve defined by the intersection* of surfaces $x^2 + 2y^2 = 3$, $x + 2y + z = 0$.