University of Notre Dame Calculus III

LECTURE 14: LOCAL MAXIMUMS AND MINIMUMS

Maxima and Minima

Definition 1. *Let $f = f(x, y)$. A point (a, b) is called*

- *local minimum if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b). $f(a, b)$ is a local minimum value of f.*
- *local maximum if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b). $f(a, b)$ is a local maximum value of f.*

A local minimum/maximum is called an absolute minimum/maximum if the respective inequalities hold for all (x, y) in the domain of f.
The procedure for finding these mirrors that in Calc I. The procedure is as follows

1. Find the critical points (a, b) of f, i.e., points (a, b) such that $\nabla f(a, b) = \vec{0}$
2. Compute the determinant of the Hessian of f at (a, b): The Hessian is

$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

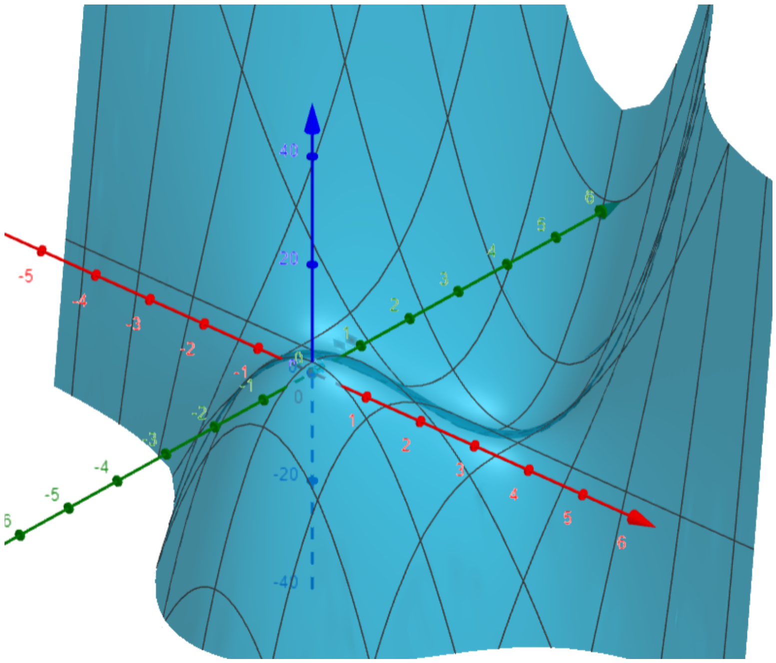
   so the number we want is

$$D(a, b) = \det Hf(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

3. Compute $f_{xx}(a, b)$ then classify using
   Second Derivatives Test:
   Suppose that f has continuous second partials near a point (a, b) such that $\nabla f(a, b) = \vec{0}$, i.e. (a, b) is a critical point of f. Let $D(a, b) = Hf(a, b)$, then:
   - if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum value of f.
   - if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum value of f.
   - if $D(a, b) < 0$, then (a, b) is a saddle point
   - if $D(a, b) = 0$, the test fails

Example 1. Find and classify the critical points of

$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$$



The graph of our function. We can plainly see that we have two critical points, both on xz-plane.

Solution:

First: $\nabla f = \langle 6xy - 12x, 3y^2 + 3x^2 - 12y \rangle$

$$\nabla f = \vec{0} = \begin{cases} 6xy - 12x = 0 \\ 3y^2 + 3x^2 - 12y = 0 \end{cases}$$

Then $6x(y-2) = 0$ implies $x = 0$ or $y = 2$.
For $x = 0$ $3y^2 - 12y = 0$ implies $3y(y-4) = 0$ implies $y = 0$ or $y = 4$. So, two critical points are $(0,0)$ and $(0,4)$
For $y = 2$ $12 + 3x^2 - 24 = 0$ implies $3x^2 = 12$ implies $x^2 = 4$ meaning $x = \pm 2$. So two more critical points are $(-2,2)$ and $(2,2)$.
Now $f_{xx} = 6y - 12$ and

$$Hf = \begin{pmatrix} 6y - 12 & 6x \\ 6x & 6y - 12 \end{pmatrix}$$

So $D = (6y-12)^2 - 36x^2$. Now we can classify:

Crit. Pt	f_{xx}	D	type
$(0,0)$	-12	144	local max
$(0,4)$	12	144	local min
$(2,2)$	0	-144	saddle
$(-2,2)$	0	-144	saddle