

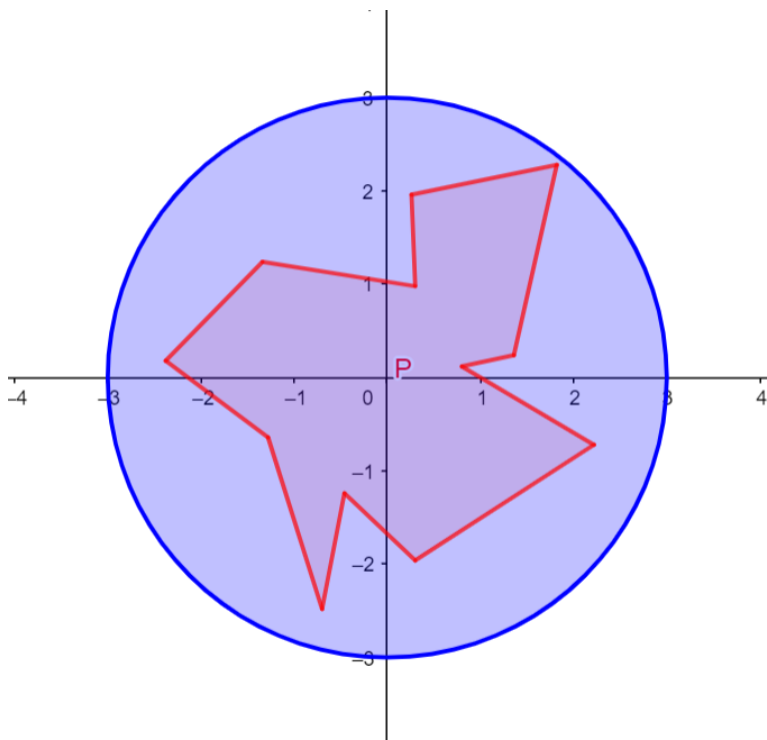
University of Notre Dame Calculus III

LECTURE 15: ABSOLUTE MAXIMUMS AND MINIMUMS

We can also ask about extremum when we restrict the domain of f . First some definitions.

Closed sets are sets which contain all of their boundary points.

A set is bounded in \mathbb{R}^2 if it can fit in a disk.



The set of points bounded by polygon P is bounded by the disc defined by all points $x^2 + y^2 \leq 9$.

An unbounded example is

$$\{(x, y) | x \geq a, b \leq y \leq c\}$$

Procedure for finding extrema on closed and bounded sets

This is the analog to finding extrema of $y = f(x)$ on $[a, b]$.

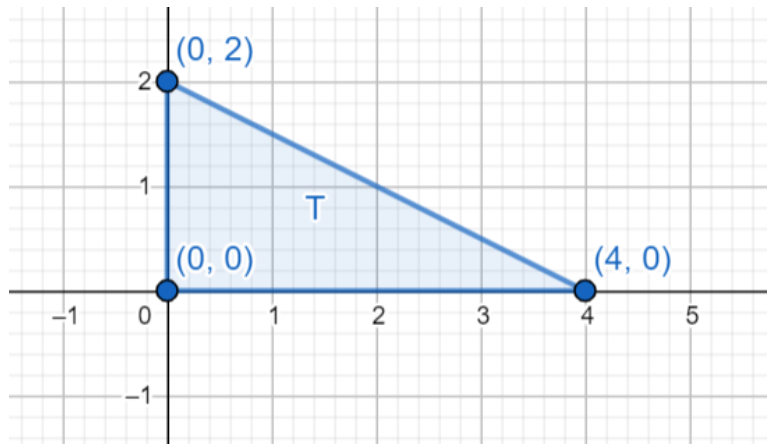
To find the absolute maximum and minimum values of a continuous function $f = f(x, y)$ on a closed and bounded set D :

1. Find the values of f at critical points of f inside D
2. Find the extreme values of f on the boundary
3. The largest value from the first two steps is the absolute max and the smallest is the absolute min

Example 1. Find the absolute maximum and minimum values of $f(x, y) = x + y - xy$ on the closed and bounded set D which is the closed triangle with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

Solution:

The region D is



$\nabla f = \langle 1 - y, 1 - x \rangle$, so the critical point of f is $(1, 1)$, which is inside D .

Now we check on the boundary of D .

L_1 : Plug $x = 0$ into f to restrict to L_1 : $g_1(y) = f(0, y) = y$. Now, treat g_1 as defined on $[0, 2]$, and do calc I. We must include the endpoints $y = 0, 2$ as possible extrema. Since $g'_1(y) = 1$, there are no extrema inside $[0, 2]$. So, L_1 gives us possible extrema at $(0, 0)$ and $(0, 2)$.

L_2 : Plug in $y = 0$: $g_2(x) = f(x, 0) = x$, $0 \leq x \leq 4$ $g'_2(x) = 1$, so we only get the endpoints as possibilities: $(0, 0)$ and $(4, 0)$.

L_3 : Plug in $y = 2 - \frac{1}{2}x$:

$$g_3(x) = f(x, 2 - \frac{1}{2}x) = x + (2 - \frac{1}{2}x) - x(2 - \frac{1}{2}x) = x + 2 - \frac{1}{2}x - 2x + \frac{1}{2}x^2 = \frac{1}{2}x^2 - \frac{3}{2}x + 2$$

Meaning $g'_3(x) = x - \frac{3}{2}$. $g'_3(x) = 0$ when $x = \frac{3}{2}$.

So there is a potential extremum at $(\frac{3}{2}, 2 - \frac{1}{2}(\frac{3}{2})) = (\frac{3}{2}, \frac{5}{4})$ as well as the endpoints $(0, 2)$ and $(4, 0)$. So all the potential points and their values are:

Point	Value of f
$(0, 0)$	0
$(4, 0)$	4
$(0, 2)$	2
$(\frac{3}{2}, \frac{5}{4})$	$\frac{3}{2} + \frac{5}{4} - (\frac{3}{2})(\frac{5}{4}) = \frac{7}{8}$
$(1, 1)$	$1 + 1 - 1 = 1$

Example 2. A cardboard box without a lid is to have a volume of 3200cm^3 . Find the dimensions of the box which uses the least amount of cardboard.

Solution:

The volume is constant $V = lwh = 32000$. We aim to minimize surface area:

$$A = lw + 2(hw + hl)$$

We only have tools to work with two variables, so we need to eliminate one, say h : Using the volume we find $h = \frac{3200}{lw}$. Note we must be careful that neither l or w are 0, but these aren't acceptable solutions. So

$$A = lw + w\left(\frac{3200}{l} + \frac{32000}{w}\right) = lw + 64000\left(\frac{1}{l} + \frac{1}{w}\right) \nabla A = \left\langle w - \frac{64000}{l^2}, l - \frac{64000}{w^2} \right\rangle$$

Now we solve $\nabla A = \vec{0}$ which tells us

$$\begin{aligned} w - \frac{64000}{l^2} &= 0 \\ l - \frac{64000}{w^2} &= 0 \end{aligned}$$

The first equation tells us $w = \frac{64000}{l^2}$ plugging this into the second we have

$$\begin{aligned} l - 64000\frac{1}{w^2} &= l - 64000\left(\frac{l^4}{(64000)^2}\right) = l - \frac{l^4}{64000} \\ &= l\left(1 - \frac{l^3}{64000}\right) = 0 \end{aligned}$$

Since we don't want $l = 0$, $l = 40$ is the only solution. When $l = 40$, $w = \frac{64000}{40^2} = 40$.

Now, we make sure it's a minimum:

$$A_{ll} = \frac{128000}{l^3} \implies A_{ll}(40, 40) = \frac{128000}{40^3} = 2\frac{40^3}{40^3} = 2 > 0$$

$$HA = \begin{pmatrix} \frac{128000}{l^3} & 1 \\ 1 & \frac{128000}{w^3} \end{pmatrix} \implies HA(40, 40) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$D(40, 40) = 4 - 1 = 3 > 0$. So $(40, 40)$ is a min. If $l = w = 40$, $V = lwh = 32000$ implies $h = 20$, so the dimensions are $40\text{cm} \times 40\text{cm} \times 20\text{cm}$

Extra Examples

1. Does the function $f = x^2 - y^2$ have any local minima or maxima?
2. Find the local min/max of $f = x^4 + y^4 - 4xy + 1$.
3. Find and classify all critical points of the function $f(x, y) = x^3 + y^3 - x - y$
4. Find the absolute maximum and absolute minimum values of

$$f(x, y) = xy - x - y$$

on the triangular region with vertices $(0, 0)$, $(2, 0)$, $(2, 4)$ and the point(s) where these extreme values are achieved.

5. Find the absolute maximum values of

$$f(x, y) = xy - x^2 + y$$

inside the region bounded by $y = x^2$ and $y = 4$ and the point(s) where these extreme values are achieved.

6. Find the absolute maximum and absolute minimum values of

$$f(x, y) = x^2 + y^2 + y$$

on the disk $x^2 + y^2 \leq 1$ and the point(s) where these extreme values are achieved.

7. Find the absolute maximum and absolute minimum values of

$$f(x, y) = (4 - y^2)e^{2x^2}$$

over the region $4x^2 + y^2 \leq 4$ and the point(s) where these extreme values are achieved.