University of Notre Dame Calculus III

LECTURE 16: EXTREMUM USING LAGRANGE MULTIPLIERS

Lagrange Multipliers

Suppose we want to extremize $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$ (this is restricting the domain of f to the constraint and finding max and min values). We'll approach this geometrically. It turns out that the smallest and largest valued level curves intersect the graph of the constraint tangently, that is, if we view the constraint as a level curve of a function $g(x,y) = k$, ($g(x,y) = -x^2 - y^2 = -1$ in this example), then ∇f is parallel to ∇g. Let's formalize this:

Say we want to extremize $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, a level surface of g. Suppose $P = (x_0, y_0, z_0)$ is an extreme point and let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a curve on the constraint surface such that $\vec{r}(t_0) = P$. We've learned that $\nabla g(P) \perp \vec{r}\,'(t_0)$. Recall that we are looking for the "highest and lowest" points on the graph of $f(x,y,z)$ over the surface $g = k$. So, since $\vec{r}(t)$ lies on $g = k$, we can plug it into f and get: $h(t) = f(\vec{r}(t))$, the values of f over $\vec{r}$. Since $\vec{r}(t_0) = P$, $h(t)$ hits an extremum at t_0 meaning that $h'(t_0) = 0$. So:

$$h'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0) = 0$$

But, this is true for all curves $\vec{r}(t)$ in $g = k$ passing through P. So, we have $\nabla f \perp g = k$ at P. But, since $\nabla g \perp g = k$ at P, that must mean $\nabla f(P)$ is parallel to $\nabla g(P)$. So $\nabla f(P) = \lambda \nabla g(P)$.

λ is called a Lagrange Multiplier.

Method of Lagrange Multipliers

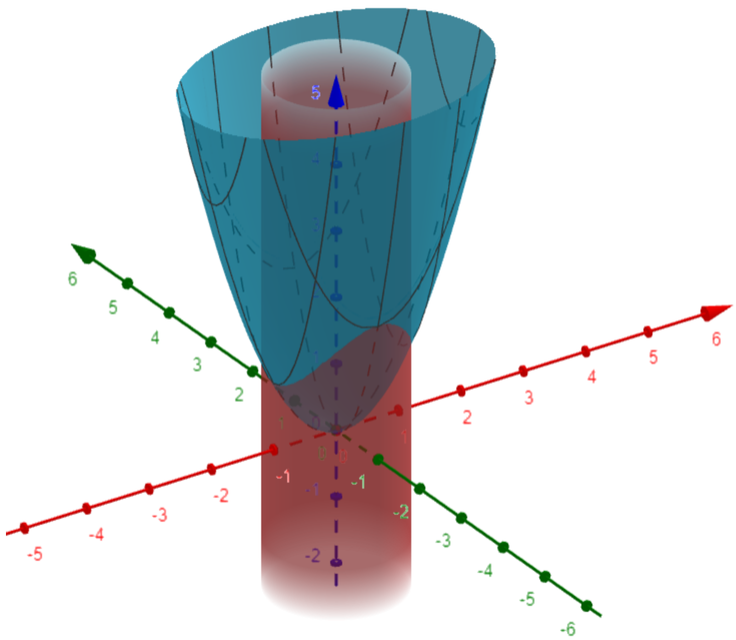
Assume that $\nabla g \neq \vec{0}$ on $g = k$ and that extreme values of $f = f(x,y,z)$ subject to $g = k$ exist. To find them

1. Find all quadruples (x, y, z, λ) solving the system:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases} \quad \Longleftrightarrow \quad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases}$$

2. Evaluate f at all the (x, y, z) generated from step 1 and identify the maxima/minima.

Example 1. Find the extreme values of $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$.



We find the minimum values of our function (blue) when constrained (or intersected) with the cylinder $x^2 + y^2 = 1$ (red).

Solution:

$g = x^2 + y^2$. $\nabla f = \langle 2x, 4y\rangle$, $\nabla g = \langle 2x, 2y\rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \implies \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

The first equation tells us $x = 0$ or $\lambda = 1$

$x = 0$ implies $y^2 = 1$ meaning $y = \pm 1$. So the candidate points are $(0,1)$ and $(0,-1)$.

$\lambda = 1$ implies from the second equation that $y = 0$ meaning $x^2 = 1$ so $x = \pm 1$. So the candidate points here are $(1,0)$ and $(-1,0)$.

All that's left now is to check

Point	Value of f
$(0,1)$	$f(0,1) = 2$
$(0,-1)$	$f(0,-1) = 2$
$(1,0)$	$f(1,0) = 1$
$(-1,0)$	$f(-1,0) = 1$

Let's see an example which also brings in ideas from the previous section*.

Example 2. Find the extreme values of $f(x, y) = e^{-xy}$ on the region $x^2 + 4y^2 \leq 1$.

Solution:

Lagrange multipliers will only find the extreme values of $f(x, y)$ when restricted to a level curve $g = k$. So, Lagrange multipliers will only work on the boundary of this region. Let's first do the interior:

$$\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$$

Since $e^{-xy} > 0$ for all (x, y) it follows that $\nabla f = \vec{0}$ only at $(0, 0)$.

Now, we check the boundary:

The boundary is $x^2 + 4y^2 = 1$, so we can see it as a level curve of $g(x, y) = x^2 + 4y^2$: $g = 1$. $\nabla g = \langle 2x, 8y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \implies \begin{cases} -ye^{-xy} = \lambda 2x \\ -xe^{-xy} = \lambda 8y \\ x^2 + 4y^2 = 1 \end{cases}$$

Let's try solving for e^{-xy}. Using the first equation we need to check what happens if $y = 0$.

If $y = 0$ then the first equation implies $\lambda = 0$ or $x = 0$, however $x = 0$ contradicts the last equation. $\lambda = 0$ if $\lambda = 0$ then $x = 0$ this again contradicts the last.

So, we have that $y = 0$ cannot happen, so $y \neq 0$.

Now we can solve the first two for λ

$$\lambda = \frac{-ye^{-xy}}{2x} = \frac{-xe^{-xy}}{8y} \qquad \implies \qquad \frac{8y}{x} = \frac{2x}{y} \qquad \implies \qquad x^2 = 4y^2$$

Now by the last equation we have $x^2 = 1 - 4y^2$, so plugging this in: $(1 - 4y^2) = 4y^2$. Meaning $y = \pm\frac{1}{2\sqrt{2}}$.

So $x^2 = 1 - 4y^2$ meaning $x = \pm\frac{1}{\sqrt{2}}$.

Thus the candidate points and values of f are:

Candidate	Value of f
$(0, 0)$	1
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{-\frac{1}{4}}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	$e^{\frac{1}{4}}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{\frac{1}{4}}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$	$e^{-\frac{1}{4}}$

This means that $e^{-\frac{1}{4}}$ is the min and $e^{\frac{1}{4}}$ is the max.

Extra Example Problems

1. Use the method of Lagrange multipliers to find the the point on the plane $2x - y + 3z + 14 = 0$ closest to the origin.

2. Find the absolute maximum and absolute minimum of

$$f(x, y) = 1 - x^2 - y^2$$

over the region $D = \left\{(x, y) : x^2 + \frac{1}{2}y^2 \leq 1\right\}$. Apply Lagrange multipliers to find absolute maximum and minimum values on the boundary $x^2 + \frac{1}{2}y^2 = 1$.

3. Find the minimum value of

$$f(x, y, z) = x^2 + y^2 + z^2$$

with respect to the constraint $x^2yz = 2$ and the point(s) where the minimum value is achieved.