University of Notre Dame Calculus III

LECTURE 17: LAGRANGE MULTIPLIERS WITH 2 CONSTRAINTS

Two Constraints

Suppose we want to extremize $f(x, y, z)$ subject to two constraints: $g(x, y, z) = c$ and $h(x, y, z) = k$. Geometrically, we are extremizing f along the curve of intersection of $g = c$ and $h = k$. Now, we still have that ∇f is perpendicular to the curve of intersection at an extreme point, P, but it isn't necessarily perpendicular to both $g = c$ and $h = k$ at this point. However, since both ∇g and ∇h are perpendicular to the curve at this point, we find that $\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$. To summarize, we now have to solve

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) = c \\ h(x, y, z) = k \end{cases}$$

Example 1. Find the points on the conic section* determined by $z^2 = x^2 + y^2$ and $z = x + y + 2$ which are closest to the origin.

Solution:

Let $g = x^2 + y^2 + z^2$ and $h = x + y - z + 2$. The function we want to minimize subject to $g = 0$ and $h = 0$ is

$$f(x, y, z) = [d((0,0,0),(x, y, z))]^2 = x^2 + y^2 + z^2$$

We can simplify our computations by minimizing d^2 instead of d. Then

$$\nabla f = \langle 2x, 2y, 2z\rangle \ , \qquad \nabla g = \langle 2x, 2y, -2z\rangle \ , \qquad \nabla h = \langle 1, 1, -1\rangle$$

$$\begin{cases} 2x = \lambda 2x + \mu \ \rightarrow \ \mu = 2x(1-\lambda) \\ 2y = \lambda 2y + \mu \ \rightarrow \ \mu = 2y(1-\lambda) \\ 2z = -\lambda 2z - \mu \\ z^2 = x^2 + y^2 \\ z = x + y + 2 \end{cases}$$

The first two together tell us either $x = y$ or $\lambda = 1$.
$\lambda = 1$: Then $\mu = 0$, so by the third equation $2z = -2z$ which implies $z = 0$. By the forth one we have $0 = x^2 + y^2$ meaning $x = y = 0$. But then plugging these values into the fifth we have $0 = 0 + 0 + 2 = 2$, which is a contradiction.
$x = y$ plugging this into the forth and fifth we have

$$\begin{cases} z^2 = 2x^2 \\ z = 2x + 2 \end{cases}$$

Plugging the later into the former, we get

$$z^2 = (2x+2)^2 = 4x^2+8x+4 = 2x^2 \qquad \Longrightarrow \qquad 2x^2+8x+4 = 0 \qquad \Longrightarrow \qquad x = \frac{-8 \pm \sqrt{64-32}}{4} = -2 \pm \sqrt{2} = y$$

We can use the later again to find z:
If $x = y = 2 + \sqrt{2}$, $z = 2(-2+\sqrt{2}) + 2 = 2\sqrt{2} - 2$
If $x = y = -2 + \sqrt{2}$, $z = 2(-2-\sqrt{2}) + 2 = -2\sqrt{2} - 2$

Candidate Points	Value
$(-2+\sqrt{2}, -2+\sqrt{2}, 2\sqrt{2}-2)$	$24 - 16\sqrt{2}$
$(-2-\sqrt{2}, -2-\sqrt{2}, -2\sqrt{2}-2$	$24 + 16\sqrt{2}$

Thus the closest point is $\left(-2+\sqrt{2}, -2+\sqrt{2}, 2\sqrt{2}-2\right)$.

Extra Examples

1. Find the Lagrange multipliers one needs to solve in order to find the minimum of the function

$$f\left(x,y,z\right)=x^2+y^2+z^2$$

on the curve of intersection of the surfaces $y^2-z^2=1$ and $x-y=1$.

2. Find the absolute maximum and absolute minimum values of

$$f(x,y,z)=2x+y$$

with respect to the constraints $g(x,y,z)=2x^2+z^2=4$ and $h(x,y,z)=2x+y+3z=6$ and the point(s) where these extreme values are achieved.