University of Notre Dame Calculus III

LECTURE 18: DOUBLE INTEGRALS OVER RECTANGULAR AND GENERAL REGIONS

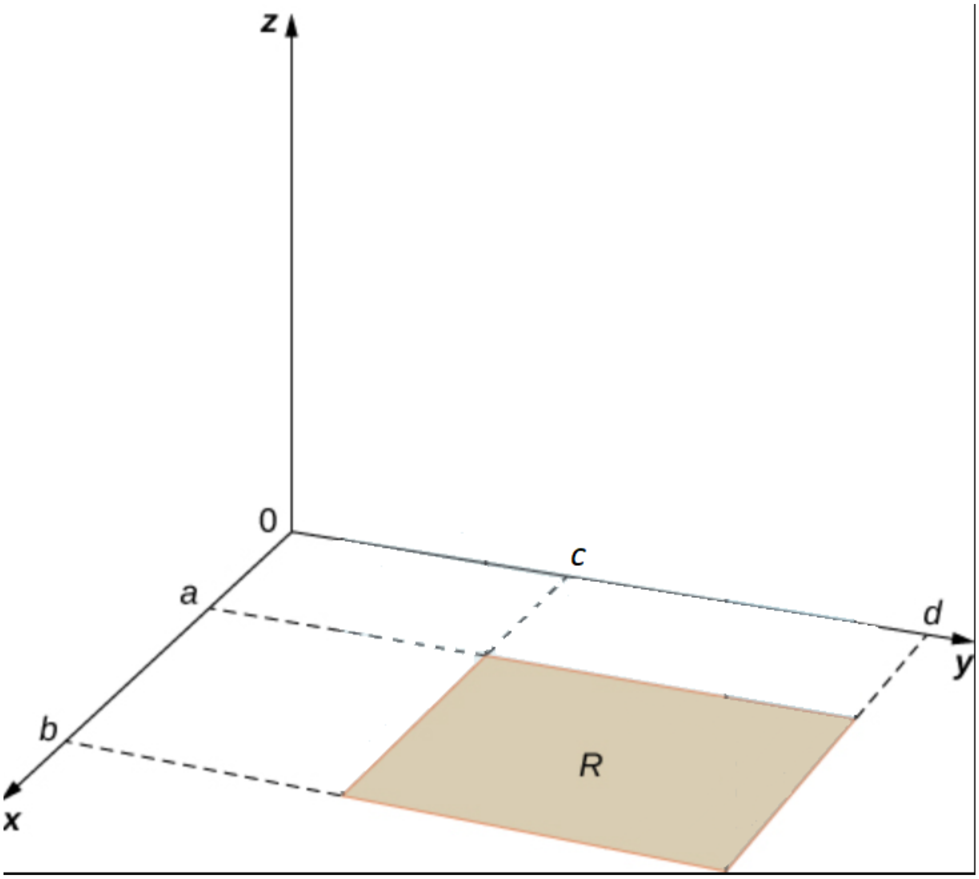
Double Integrals Over Rectangles

Let's recall how integrals are defined: An integral is the signed area under a curve; put mathematically,

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$
$$\text{Area} = \int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

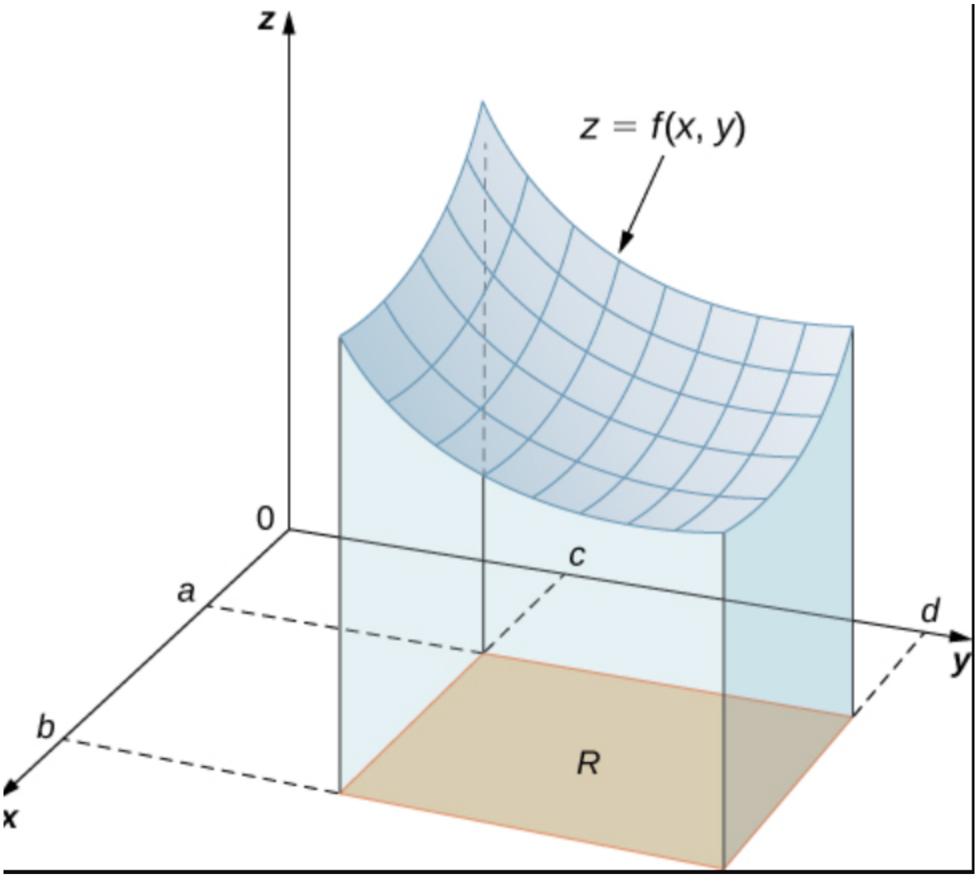
Double Integrals

Let's start with a simple region: a rectangle.



Let $R = [a,b]\times[c,d]$ and let $f = f(x,y)$ contain R in its domain. We'll also assume for now that $f \geq 0$ on R. We start by cutting up the rectangle

In each subrectangle R_{ij} we choose a simple point (x_{ij}^*, y_{ij}^*), and over each R_{ij} construct a column of height $f(x_{ij}^*, y_{ij}^*)$.



Adding up these volumes gives an approximation of the volume under f:

$$\text{Vol} \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\left(f(x_{ij}^{*},y_{ij}^{*})\Delta x\Delta y\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*},y_{ij}^{*})\Delta A$$

This is the double Riemann sum.
Now, of course, to get the actual volume we have to take finer and finer partitions ($\Delta x, \Delta y \to 0 \iff m, n \to \infty$)
So

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*},y_{ij}^{*})\Delta A$$

Now, none of this required $f \geq 0$ on R, so we get the final definition:

Definition 1. *The double integral of f over the rectangle R is*

$$\int\int_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*},y_{ij}^{*})\Delta A$$

if the limit exists.

Facts

1. If $f(x,y) \geq 0$, then the volume V of the solid which lies above R and below the surface $z = f(x,y)$ is

$$V = \int\int_R f(x,y)\,dA$$

2.
$$\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA$$

3.
$$\int\int_R cf(x,y)\,dA = c\int\int_R f(x,y)\,dA$$

4. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\int\int_R f(x,y)\,dA \geq \int\int_R g(x,y)\,dA$$

Iterated Integrals

This will give us a way of actually computing double integrals. Using the definition, let's take the "m-limit" first. This amounts to integrating x first:

$$\begin{aligned}\int\int_R f(x,y)\,dA &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta x\right)\Delta y\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\int_a^b f(x,y_j^*)\,ddx\right)\Delta y\\ &= \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy\\ &= \int_c^d\int_a^b f(x,y)\,dx\,dy\end{aligned}$$

Doing the "n-limit" first would give integrating y first

$$\int\int_r f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx\,.$$

These are called iterated integrals. Notice that we are working from the inside out in these integrals. So, now we have the question of how to compute $\int_a^b f(x,y)\,dx$ or $\int_c^d f(x,y)\,dy$? The answer is partial integration which are performed analogusly to partial derivatives.

Example 1. Compute $\int_0^5 12x^2y^3\,dx$ and $\int_0^1 12x^2y^3\,dy$ and the associated indefinite integrals.

Solution:

First

$$\int 12x^2y^3\,dx = 4x^3y^3 + g(y)$$

Notice that the "constant" term in this indefinite integral is a function of y because "x sees y" as a constant.

$$\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 4\cdot 5^3\cdot y^3 = 500y^3$$
$$\int 12x^2y^3\,dy = 3x^2y^4 + h(x)$$
$$\int_0^1 12x^2y^3\,dy = 3x^2y^4\Big|_0^1 = 3x^2\cdot 1^4 - 0 = 3x^2$$

Example 2. Compute

$$\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx \qquad \int_0^4\int_0^2 y^3e^{2x}\,dx\,dy$$

Solution:

$$\begin{aligned}\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx &= \int_0^2 \frac{1}{4}y^3e^{2x}\Big|_0^4\,dx\\ &= \int_0^2 64e^{2x}\,dx\\ &= 32e^{2x}\Big|_0^2 = 32(e^4-1)\end{aligned}$$

$$\begin{aligned}\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy &= \int_0^4 \frac{1}{2}y^3e^{2x}\Big|_0^2\,dy\\ &= \int_0^4 \frac{1}{2}(e^4-1)y^3\,dy\\ &= \frac{1}{8}(e^4-1)y^3\Big|_0^4\\ &= \frac{1}{8}(e^4-1)\cdot 256 = 32(e^4-1)\end{aligned}$$

These integrals being equal is no coincidence:

Theorem 2. *Fubini's Theorem If f is continuous on $R = [a,b]\times[c,d]$ then*

$$\int\int_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

(There are more general conditions f could satisfy: f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the integral exists.)

Example 3. Compute

$$\int\int_R ye^{-xy}\,dA$$

where $R = [0,2] \times [0,3]$.

Solution:

Since this looks annoying to integrate y first, let's integrate w.r.t. x first. Then

$$\begin{aligned}\int\int_R ye^{-xy}\,dA &= \int_0^3\int_0^2 ye^{-xy}\,dx\,dy = \int_0^3 y\left(\frac{-1}{y}e^{-xy}\right)\Big|_0^2\,dy\\ &= \left(y + \frac{1}{2}e^{-2y}\right)\Big|_0^3 = \left(3 + \frac{1}{2}e^{-6}\right) - \left(0 + \frac{1}{2}\right)\\ &= \frac{5}{2} + \frac{1}{2}e^{-6}\end{aligned}$$

Let's end with a volume example:

Example 4. Find the volume of the solid bounded by the surface $z = 1 + e^x \sin y$ and the planes $x = 1$, $x = -1$, $y = 0$, $y = \pi$, and $z = 0$.

Solution:

Since $e^x > 0$ and $\sin y \geq 0$ on $0 \leq y \leq \pi$, we have $z \geq 0$. The base of this solid is $R = [-1,1] \times [0,\pi]$ and its height is z. Thus,

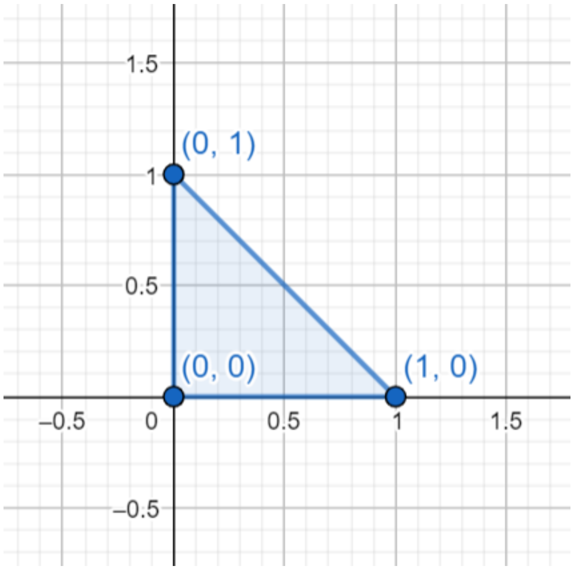
$$\begin{aligned}\text{Vol} &= \int\int_R \left(1 + e^x \sin y\right)\,dA = \int_{-1}^1\int_0^\pi \left(1 + e^x \sin y\right)\,dy\,dx\\ &= \int_{-1}^1 \left(y - e^x \cos y\right)\Big|_0^\pi\,dx = \int_{-1}^1 \left[(\pi + e^x) - (-e^x)\right]\,dx\\ &= \int_{-1}^1 \left(2e^x + \pi\right)\,dx = \left(2e^x + \pi x\right)\Big|_{-1}^1 = (2e + \pi) - (2e^{-1} - \pi)\\ &= 2e - 2e^{-1} + 2\pi\end{aligned}$$

Extra Problems

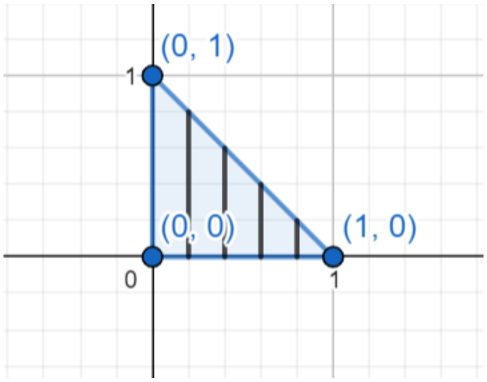
1. Evaluate the iterated integral $\int_0^3\int_1^2 x^2 y\,dy\,dx$.
2. Evaluate the area integral $\iint_R (x - 3y^2)\,dA$ where $R = [0,2] \times [1,2]$.
3. Find the average of $f = e^y\sqrt{x + e^y}$ over the rectangle $[0,4] \times [0,1]$.

Double Integrals over General Regions

Let's consider the problem of integrating the function $z = 5$ over the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$



So, we're computing $\int\int_D 5\,dA$ where D is the triangular region. How do we compute this? Slices Recall that to perform a double integral, we compute the inner integral first by holding one variable constant and integrating w.r.t. the other. Let's say we integrate w.r.t. y first. This means that we fix x and integrate from the smallest y-value to the largest at this x-value:



"vertical slices"

So, the bounds on the y-integral are $0 \leq y \leq 1 - y$. Now, all we have to do is add up over all the possible y-values. So

$$\int_0^1 \int_0^{1-x} 5\,dy\,dx = \int 015y\Big|_0^{1-x} dx = \int_0^1 5 - 5x\,dx$$

$$= \left(5x - \frac{5}{2}x^2\right)\Big|_0^1 = 5 - \frac{5}{2}$$

$$= \frac{5}{2}$$

Comments:

1. The outside integral should NEVER have a variable in it!
2. Always sketch the region of integration. It really helps when setting up bounds.
3. To find bounds, first sketch the region, then decide which to take slices:
    - vertical (holding x constant first): look from the bottom to top:

$$\int\int_D f(x,y)\,dA = \int_a^b \int^d g\,d(x)h(x)f(x,y)\,yx$$

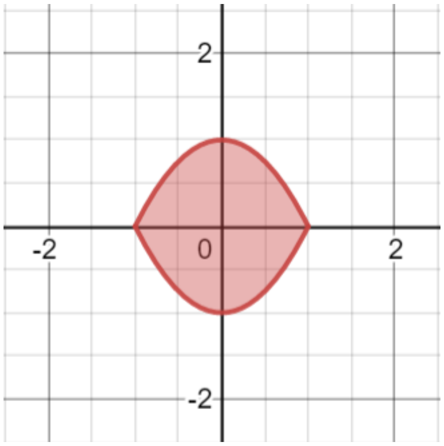
    - horizontal (holding y constant first) look from left to right:

$$\int\int_D f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

4. Sometimes integrating one way is easier than another

Example 5. Compute $\int\int_D x\,dA$ where D is the region bounde by the parabolas $y = 1-x^2$ and $y = x^2-1$

Solution:
Step 1: Sketch the region!



Notice that vertical slices work better here since to do horizontal would require splitting the integral into two pieces (also the bounds wouldn't be as nice).
Step 2: Set up the integral

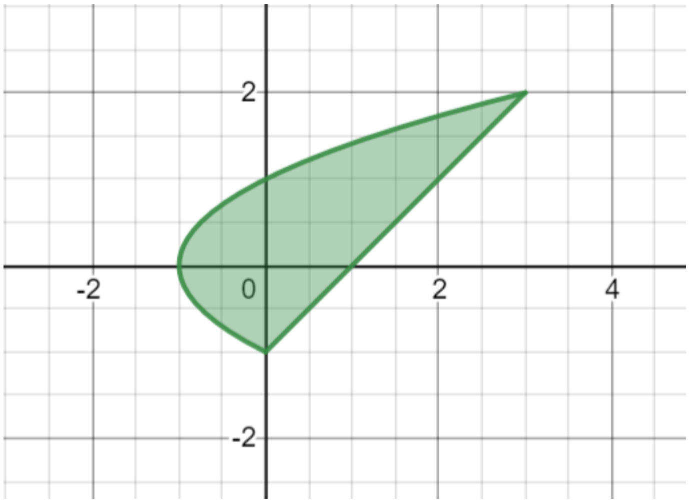
$$\int\int_D x\,dA = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} x\,dy\,dx = \int_{-1}^{1} xy\Big|_{x^2-1}^{1-x^2} dx$$
$$= \int_{-1}^{1} x\left[(1-x^2)-(x^2-1)\right] dx = \int_{-1}^{1} 2x - 2x^3\,dx$$
$$= \left(x^2 - \frac{1}{2}x^4\right)\Big|_{-1}^{1} = \left(1-\frac{1}{2}\right) - \left(1-\frac{1}{2}\right) = 0$$

From now on, we'll focus more on setting up integrals.

Example 6. Set up the integral to find the area of the region bounded by $x = y^2 - 1$ and $y = x - 1$

hrule Notice that the volume of an object with height 1 is equal to the area of its base (ignoring units, of course). So, area of $D = A(D) = \int\int_D 1\, dA = \int\int_D dA$

Solution:
First, sketch the region:



Horizontal slices will work more nicely this time. The left function is $x = y^2 - 1$ and the right function is $x = y + 1$. Now, we just need the bounds on y. We find the max and min values of y. Plugging $x = y^2 - 1$ into $y = x - 1$ gives

$$y = y^2 - 2 \quad \Longrightarrow \quad y^2 - y - 2 = (y-2)(y+1) = 0 \quad \Longrightarrow \quad y = -1, 2$$

Thus:

$$\text{Area of } D = A(D) = \int\int_D dA = \int_{-1}^{2}\int_{y^2-1}^{y+1} dA$$

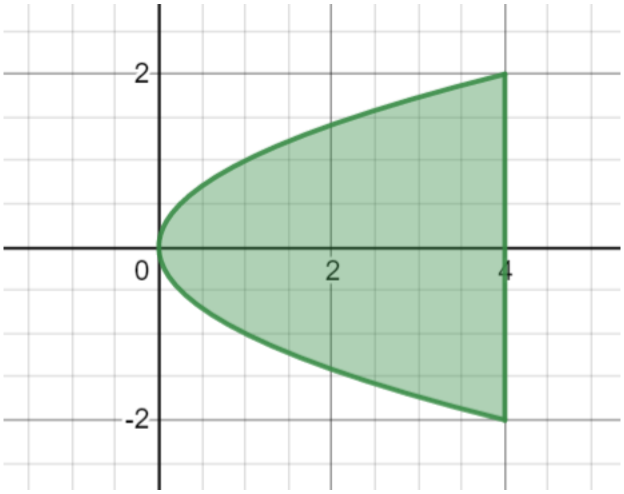
You also need to be able to read the region of integration off of a double integral:

Example 7. Compute $\int_0^2 \int_{y^2}^4 y\cos(x^2)\,dx\,dy$.

Solution:
As it stands we cannot compute it since we cannot compute $\int \cos(x^2)\,dx$... However, we can try to switch the order in which we integrate to $dy\,dx$. To do this, sketch the region:

- The outer integral says
$$0 \le y \le 2$$
- x goes between y^2 and 4 for any fixed y, so
$$y^2 \le x \le 4\,.$$

We can then draw the region



So, rewriting the integral we have

$$\begin{aligned}\int_0^2 \int_{y^2}^4 y\cos(x^2)\,dx\,dy &= \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx \\ &= \int_0^4 \frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{1}{2}x\cos(x^2)\,dx \\ &= \int_0^{16} \frac{1}{4}\cos(u)\,du = \frac{1}{4}\sin u\Big|_0^{16} = \frac{1}{4}\sin 16\end{aligned}$$

Last comment: If $D = D_1 \cup D_2$, then

$$\int\int_D f(x,y)\,dA = \int\int_{D_1} f(x,y)\,dA + \int\int_{D_2} f(x,y)\,dA$$

that is

Sometimes there's no choice but to split the integral in pieces.

Extra Problems

1. Compute the integral $\iint_D (x+2y)\,dA$ where D is bounded region between the parabolas $y = 2x^2$, $y = 1 + x^2$.
2. Let E be the tetrahedron with vertices $(0,0,0), (1,0,0), (0,2,0), (0,0,2)$. Find the volume of E.
3. Reverse order of integration in $\int_{-3}^{3}\int_{1}^{10-x^2} dy\,dx$.
4. Evaluate the integral $\int_0^1 \int_{3y}^{3} e^{x^2}\,dx\,dy$ by reversing the integral.