University of Notre Dame Calculus III

LECTURE 19:

Double Integrals in Polar Coordinates

Consider the integral $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk. How can we compute it? The answer is polar coordinates. Let's practice describing regions in polar coordinates.

Example 1. Describe the following regions in polar coordinates

1. $D = \{(r,\theta)|r \le 1\} = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$
2. $D = \{(r,\theta)|1 \le r \le 2, 0 \le \theta \le \pi\}$
3. $D = \{(r,\theta)|r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$

These types of regions are called polar rectangles since if you graph them in the r,θ-plane, they're rectangles. The most general polar rectangle is a sector of the form

$$D = \{(r,\theta)|r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}, \qquad (0 \le \theta_2 \le \theta_1 \le 2\pi)$$

The area of D is

$$\begin{aligned} A &= \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2+r_1)(r_2-r_1)\Delta\theta \\ &= r^*\Delta r\Delta\theta \end{aligned}$$

where $r^* = \frac{1}{2}(r_1+r_2)$. This tells us $dA = r\,dr\,d\theta$. This means we can change from Cartesian to polar, we have

$$\int\int_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 2. Compute $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk

Solution:
D in polar coordinates is $\{(r,\theta)|r \le 1\}$. The integrand becomes e^{r^2} because $r^2 = x^2 + y^2$, so we get

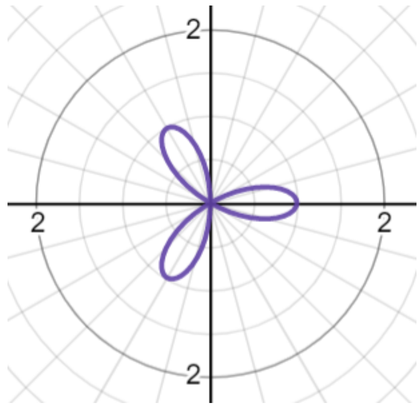
$$\begin{aligned} \int\int_D e^{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}e^u\,du\,d\theta \\ &= \int_0^{2\pi}\frac{1}{2}(e-1)\,d\theta = \pi(e-1) \end{aligned}$$

Regions of integration need not be polar rectangles. Consider the following problem from Calc II:

Example 3. Find the area enclosed by one petal of the rose

$$r = \cos 3\theta$$

Solution:
The rose looks like



We know $\cos 3\theta = 0$ when $3\theta = \frac{\pi}{2} + n\pi$ iff $\theta = \frac{\pi}{6} + \frac{n\pi}{3}$. Taking $\frac{-\pi}{6} \leq \theta \leq \frac{\pi}{6}$, we get the indicated petal. So to get the bounds on r we fix a θ-value (a ray coming out of the origin) and find an "inner" and "outer" bound on r. In this case, $0 \leq r \leq \cos 3\theta$. So

$$\begin{aligned}
A(D) &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\, dr\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}(1+\cos 6\theta)\, d\theta \\
&= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right) - \left(\frac{-\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right] \\
&= \frac{\pi}{12}
\end{aligned}$$

Example 4. Set up an integral giving the volume of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$.

Solution:
First, we rewrite these in polar coordinates paraboloid: $z = 4 - r^2$, cylinder: $r^2 = 2r\sin\theta$ iff $r = 2\sin\theta$.
In the cylinder, $r = 0$ iff $2\sin\theta = 0$ iff $\theta = \pi + n\pi$.
So, letting θ go from 0 to π, we get the cylinder. The bounds on r go from 0 to $2\sin\theta$. Thus

$$\begin{aligned}
\text{Vol} &= \int\int_D z\, dA = \int_0^{\pi}\int_0^{2\sin\theta} (4-r^2) r\, dr\, d\theta = \int_0^{\pi}\int_0^{2\sin\theta} (4r - r^3)\, dr\, d\theta \\
&= \int_0^{\pi} (8\sin^2\theta - 4\sin^4\theta)\, d\theta = \int_0^{\pi} 2\sin^2\theta(4 - 2\sin^2\theta)\, d\theta = \int_0^{\pi} (1-\cos 2\theta)(4-(1-\cos 2\theta)\, d\theta \\
&= \int_0^{\pi} \left(3 - 2\cos 2\theta - \cos^2 2\theta\right)\, d\theta = \int_0^{\pi}\left(\frac{5}{2} - 2\cos 2\theta - \frac{1}{2}\cos 4\theta\right) d\theta = \frac{5\pi}{2}
\end{aligned}$$

Extra Problems

1. Evaluate $\iint_R (3x+4y^2)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$, $x^2+y^2=4$.

2. Find the double integral $\iint_D \sqrt{1-y^2}\,dA$ where $D=\{x\geq 0, y\geq 0, x^2+y^2\leq 1\}$.

3. Evaluate the integral $\iint_D e^{-x^2-y^2}\,dA$ by changing to polar coordinates, where $D=\{(x,y): x^2+y^2\leq 1\}$.