University of Notre Dame Calculus III

LECTURE 19:

Double Integrals in Polar Coordinates

Consider the integral $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk. How can we compute it? The answer is polar coordinates. Let's practice describing regions in polar coordinates.

Example 1. Describe the following regions in polar coordinates

1. $D = \{(r,\theta)|r \le 1\} = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\}$
2. $D = \{(r,\theta)|1 \le r \le 2, 0 \le \theta \le \pi\}$
3. $D = \{(r,\theta)|r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$

These types of regions are called polar rectangles since if you graph them in the r,θ-plane, they're rectangles. The most general polar rectangle is a sector of the form

$$D = \{(r,\theta)|r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}, \qquad (0 \le \theta_2 \le \theta_1 \le 2\pi)$$

The area of D is

$$\begin{aligned} A &= \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2+r_1)(r_2-r_1)\Delta\theta \\ &= r^*\Delta r\Delta\theta \end{aligned}$$

where $r^* = \frac{1}{2}(r_1+r_2)$. This tells us $dA = r\,dr\,d\theta$. This means we can change from Cartesian to polar, we have

$$\int\int_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

Example 2. Compute $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk

Solution:

Regions of integration need not be polar rectangles. Consider the following problem from Calc II:

Example 3. Find the area enclosed by one petal of the rose

$$r = \cos 3\theta$$

Solution:

Example 4. Set up an integral giving the volume of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$.

Solution:

Extra Problems

1. Evaluate $\iint_R (3x + 4y^2)\,dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.

2. Find the double integral $\iint_D \sqrt{1 - y^2}\,dA$ where $D = \{x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$.

3. Evaluate the integral $\iint_D e^{-x^2-y^2}\,dA$ by changing to polar coordinates, where $D = \{(x, y) : x^2 + y^2 \leq 1\}$.