University of Notre Dame Calculus III

LECTURE 20: INTRODUCTION TO TRIPLE INTEGRALS

Triple Integrals

As you might imagine, triple integrals are defined using a triple Riemann sum. We'll leave the details of that to the book.
Let's start with the most basic example:

Example 1. Compute the triple integral of $f(x,y,z) = x^2ye^{xyz}$ over the box $B = [0,1] \times [1,2] \times [2,3]$.

Solution:

$$\begin{aligned}
\int\int\int_B x^2ye^{xyz}\,dV &= \int_0^1\int_1^2\int_2^3 x^2ye^{xyz}\,dz\,dy\,dx \\
&= \int_0^1\int_1^2 xe^{xyz}\Big|_2^3\,dy\,dx = \int_0^1\int_1^2 \left(xe^{3xy} - xe^{2xy}\right)\,dy\,dx \\
&= \int_0^1 \left(\frac{1}{3}e^{3xy} - \frac{1}{2}e^{2xy}\right)\Big|_1^2\,dx = \int_0^1 \left[\left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x}\right) - \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)\right]\,dx \\
&= \int_0^1 \left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x} - \frac{1}{3}e^{3x} + \frac{1}{2}e^{2x}\right)\,dx \\
&= \left(\frac{1}{18}e^{6x} - \frac{1}{8}e^{4x} - \frac{1}{9}e^{3x} + \frac{1}{4}e^{2x}\right)\Big|_0^1 \\
&= \frac{e^6}{18} - \frac{e^4}{8} - \frac{e^3}{9} + \frac{e^2}{4} - \left(\frac{1}{18} - \frac{1}{8} - \frac{1}{9} + \frac{1}{4}\right)
\end{aligned}$$

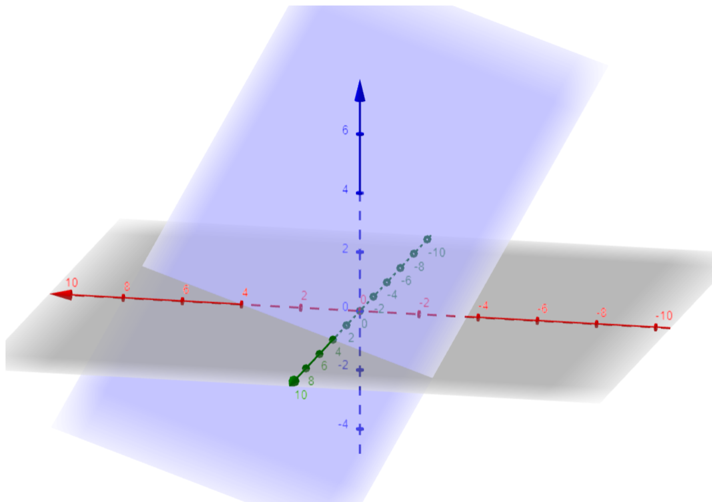
There is, of course, no reason to stick to boxes.
Let's say our region is E. When setting up bounds, they look as follows:
x: "back to front"
y: "left to right"
z: "bottom to top"
Again, sketching the region will be important! Now, once we've figured out the bounds on the inside integral, the outer two integrals' bounds come from setting up a double integral over a "shadow" region:

If the inside integral is respect to $\begin{matrix} x \\ y \\ z \end{matrix}$, then we look at the shadow of E in the $\begin{matrix} yz \\ xy \\ xz \end{matrix}$ -plane and set up the double integral over that.
Let's make this concrete with an example.
Example 2. Set up the integral to compute the volume of E, where E is the tetrahedron bounded by the

planes $x = 0$, $y = 0$, $z = 2$, and $x + y + z = 4$.

Solution:

Begin by sketching;



Now, we need an order of integration. Let's integrate y first... because, why not?

So we look from left to right and see that the left function is $y = 0$ and the right function is $x + y + z = 4 \iff y = 4 - x - z$. So the inside integral is $\int_0^{4-x-z} dV$. The shadow E makes in the xz-plane is the triangle with vertices $(0,0), (0,2), (2,0)$.

So, the volume is:

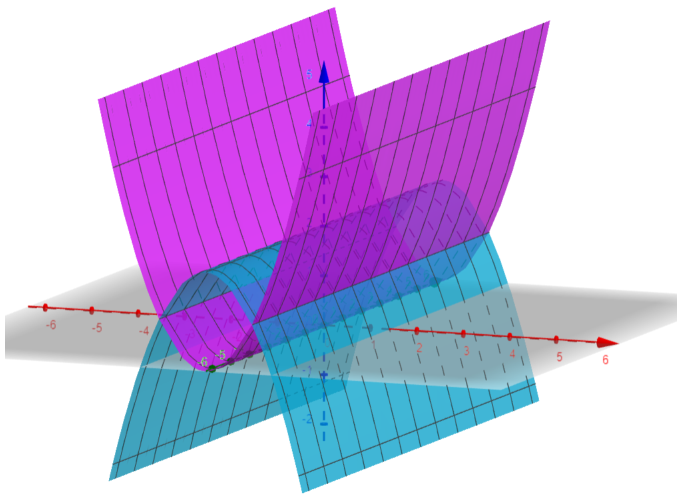
$$\text{Vol}(E) = \int\int\int_E dV = \int_0^2 \int_0^{4-x} \int_0^{4-x-z} dy\, dz\, dx$$

As with double integrals, we may need to switch the order of integration.

Example 3. Rewrite $\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy$ using $dy\,dz\,dx$.

Solution:

We begin by sketching the region:



It's simple to see here that y goes from 0 to 4. Now, the shadow in the xz-plane is:
So,

$$\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy = \int_{-1}^1 \int_{x^2}^{2-x^2} \int_0^4 xyz\,dy\,dz\,dx$$

Now for an application let's take a detour.

Extra Problems

1. Calculate the triple integral $\iiint_E 6xy\,dV$, where E is the solid region

$$E = \{(x, y, z) \mid 0 \le y \le 1,\ y \le x \le 2y,\ 0 \le z \le x + y\}.$$

2. Let E be the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$. What is the projection of E to the xy-plane? Describe E as a solid region of type 1, i.e., in the form

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

where D is a region in the xy-plane, and u_1, u_2 are suitable functions on D.

3. Compute the volume of tetrahedron E of the problem above.

4. Express the integral $\iiint_E f(x, y, z)\,dV$ as an iterated integral in different ways, where E is the solid bounded by the plane $y = 0$ and the paraboloid $y = 4 - x^2 - 4z^2$.