University of Notre Dame Calculus III

LECTURE 20: INTRODUCTION TO TRIPLE INTEGRALS

Triple Integrals

As you might imagine, triple integrals are defined using a triple Riemann sum. We'll leave the details of that to the book.
Let's start with the most basic example:

Example 1. Compute the triple integral of $f(x,y,z) = x^2 ye^{xyz}$ over the box $B = [0,1] \times [1,2] \times [2,3]$.

Solution:

There is, of course, no reason to stick to boxes.
Let's say our region is E. When setting up bounds, they look as follows:
$\underline{x}$: "back to front"
$\underline{y}$: "left to right"
$\underline{z}$: "bottom to top"
Again, sketching the region will be important! Now, once we've figured out the bounds on the inside integral, the outer two integrals' bounds come from setting up a double integral over a "shadow" region: If the inside integral is respect to $\begin{matrix} x \\ y \\ z \end{matrix}$, then we look at the shadow of E in the $\begin{matrix} yz \\ xy \\ xz \end{matrix}$-plane and set up the double integral over that.
Let's make this concrete with an example.
Example 2. Set up the integral to compute the volume of E, where E is the tetrahedron bounded by the

planes $x = 0$, $y = 0$, $z = 2$, and $x + y + z = 4$.

Solution:

As with double integrals, we may need to switch the order of integration.

Example 3. Rewrite $\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy$ using $dy\,dz\,dx$.

Solution:

Now for an application let's take a detour.

Extra Problems

1. Calculate the triple integral $\iiint_E 6xy\,dV$, where E is the solid region

$$E = \{(x, y, z) \mid 0 \le y \le 1,\ y \le x \le 2y,\ 0 \le z \le x + y\}.$$

2. Let E be the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$. What is the projection of E to the xy-plane? Describe E as a solid region of type 1, i.e., in the form

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

   where D is a region in the xy-plane, and u_1, u_2 are suitable functions on D.

3. Compute the volume of tetrahedron E of the problem above.

4. Express the integral $\iiint_E f(x, y, z)\,dV$ as an iterated integral in different ways, where E is the solid bounded by the plane $y = 0$ and the paraboloid $y = 4 - x^2 - 4z^2$.