

University of Notre Dame Calculus III

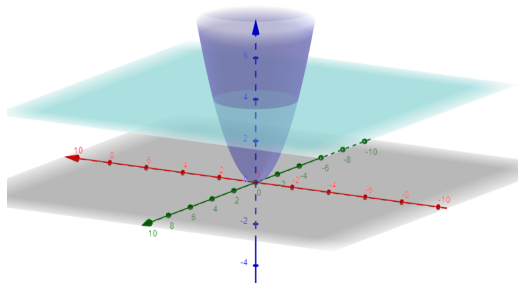
LECTURE 21: TRIPLE INTEGRALS USING CYLINDRICAL COORDINATES

Cylindrical Coordinates

Example 1. Compute the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

Solution:

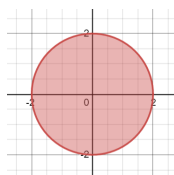
We start by sketching the region:



It looks easiest to start by integrating z first

$$\text{Vol} = \int_{?}^{?} \int_{?}^{?} \int_{x^2+y^2}^4 dz dr d\theta$$

The shadow of this region in the xy -plane is just the disk of radius 2:



We could set up bounds in cartesian here, but since this is a disk, polar coordinates are easiest. The function we're integrating over the disk is

$$f(x, y) = \int_{x^2+y^2}^4 dz \iff f(r \cos \theta, r \sin \theta) = \int_{r^2}^4 dz$$

So we should have

$$\begin{aligned} \text{Vol} &= \int_0^{2\pi} \int_0^2 \left(\int_{r^2}^4 dz \right) dr d\theta = \int_0^{2\pi} \int_0^2 z \Big|_{r^2}^4 r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 (4 - r - r^3) dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_0^2 d\theta \\ &= \int_0^{2\pi} (8 - 4) d\theta = 8\pi \end{aligned}$$

Notice how this integral was done in polar coordinates, but with z just tacked on. This is exactly what cylindrical coordinates are:

$$(r, \theta, z) \quad \text{where} \quad x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

As z is just tacked on, we find

$$dV = r dr d\theta dz$$

Cylindrical coordinates can help with triple integrals as polar did with double, as the previous example shows.

Example 2.

- a) Write the point with cylindrical coordinates $(4, \frac{\pi}{3}, -2)$ in cartesian coordinates
- b) Write the point with cartesian coordinates $(-2, 2\sqrt{3}, 3)$ in cylindrical coordinates.

Solution:

a)

$$(x, y, z) = (r \cos \theta, r \sin \theta, z) = \left(4 \cos \frac{\pi}{3}, 4 \sin \frac{\pi}{3}, -2\right) = (2, 2\sqrt{3}, -2)$$

b) $r^2 = x^2 + y^2 = 4 + 12 = 16$ implies $r = 4$

$$\tan \theta = \frac{y}{x} = -\sqrt{3} \iff \theta = \frac{2\pi}{3} + n\pi$$

Take $\theta = \frac{2\pi}{3}$ since $(-2, 2\sqrt{3})$ is in quadrant II. So

$$(r, \theta, z) = \left(4, \frac{2\pi}{3}, 3\right)$$

Extra Problems

- 1. Describe the surface whose equation in cylindrical coordinates is $z = r$.
- 2. Let $D \subset \mathbb{R}^2$ be the region given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

and let $E \subset^3$ be the solid region given by

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Write the integral $\iiint_E f(x, y, z) dV$ as a triple integral in cylindrical coordinates.

- 3. Evaluate $\iiint_E \sqrt{x^2 + y^2} dV$, where E is the region between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.