University of Notre Dame Calculus III

LECTURE 22: TRIPLE INTEGRALS USING SPHERICAL COORDINATES

Spherical Coordinates

As we can use cylinders to give coordinates on $\mathbb{R}^3$ we can also use spheres. These coordinates are obtained by rotating polar coordinates into $\mathbb{R}^3$. Spherical coordinates are (ρ, θ, ϕ) where ρ is the distance from the origin, θ is the angle made with the positive x-axis in the xy-plane, and ϕ is the angle made with the positive z-axis. So, we have

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi \;.$$

The relation to cartesian is

$$\boxed{x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\phi}$$

We also have

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Example 1.

a) Write the point with spherical coordinates $\left(3, \frac{\pi}{2}, \frac{3\pi}{4}\right)$ in cartesian coordinates.

b) Write the point with cartesian coordinates $(-1, 1, -\sqrt{2})$ in spherical coordinates.

Solution:

a)

$$x = \rho\cos\theta\sin\phi = 3\cos\frac{\pi}{2}\sin\frac{3\pi}{4} = 3\cdot 0\cdot\sqrt{2}2 = 0$$
$$y = \rho\sin\theta\sin\phi = 3\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 3\cdot 1\cdot\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$
$$z = \rho\cos\phi = 3\cos\frac{3\pi}{4} = \frac{-3\sqrt{2}}{2}$$

So $(x, y, z) = \left(0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$

b)

$$\rho^2 = x^2 + y^2 + z^2 = 1 + 1 + 2 = 4 \qquad \Longrightarrow \rho = 2$$
$$z = \rho\cos\phi \Longleftrightarrow -2\sqrt{2} = 2\cos\phi \Longrightarrow \cos\phi = -\frac{\sqrt{2}}{2} \Longrightarrow \phi = \frac{3\pi}{4}$$
$$y = \rho\sin\theta\sin\phi \Longleftrightarrow 2\sin\theta\sin\frac{3\pi}{4} = \sqrt{2}\sin\theta \Longrightarrow \sin\theta = \frac{\sqrt{2}}{2} \Longrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Checking with x:

$$x = \rho\cos\theta\sin\phi \Longleftrightarrow -1 = 2\cos\theta\sin\frac{3\pi}{4} = \sqrt{2}\cos\theta$$

So $\cos\theta \leq 0$ implies $\theta = \frac{3\pi}{4}$

Thus, $(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Example 2. Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4z$ and above the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$.

Solution:

Let's rewrite the equations in spherical coords

$$\rho^2 = x^2 + y^2 + z^2 = 4z = 4\rho\cos\phi$$

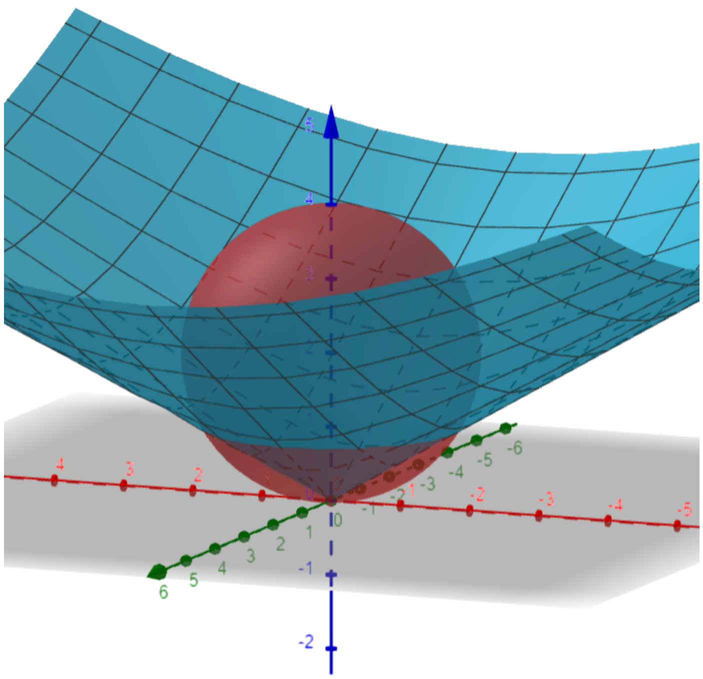
So $\phi = 4\cos\phi$

$$\rho\cos\phi = z = \sqrt{\frac{1}{3}\left(x^2+y^2\right)} = \sqrt{\frac{1}{3}\left(\rho^2\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right)} = \sqrt{\frac{1}{3}\rho^2\sin^2\phi}$$
$$= \frac{1}{\sqrt{3}}\rho\sin\phi \Longrightarrow \cos\phi = \frac{1}{\sqrt{3}}\sin\phi \Longrightarrow \tan\phi = \sqrt{3} \Longrightarrow \phi = \frac{\pi}{3}$$

Rewriting the sphere in standard form gives

$$x^2 + y^2 + (z-2)^2 = 4$$

a sphere of radius 2 centered at $(0,0,2)$. The cone is given by $\phi = \frac{\pi}{3}$, so a sketch of the region is



We end up with a "snow cone" type object. Then the volume is

$$\text{Vol} = \int\int\int_E dV = \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{4\cos\phi} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\frac{64}{3}\cos^3\phi\sin\phi\,d\theta\,d\phi = \frac{128\pi}{3}\int_0^{\frac{\pi}{3}}\cos^3\phi\sin\phi\,d\phi$$
$$= \frac{32\pi}{3}(-\cos^4\phi)\Big|_0^{\frac{\pi}{3}} = \frac{32\pi}{3}\left(-\frac{1}{16}-(-1)\right) = 10\pi$$

Extra Problems

1. Describe the surfaces whose equations in cylindrical coordinates are $(a)\, \rho = c$, $(b)\, \theta = c$, $(c)\, \phi = c$.

2. Suppose R is the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, under the cone $z = \sqrt{x^2 + y^2}$, and above the cone $z = -\sqrt{x^2 + y^2}$. Write the following triple integral in spherical coordinates

$$\iiint_R z^2 \, dV.$$

3. Suppose E is the region bounded by the spheres $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 9$ and above the cone $\phi = \pi/3$. Evaluate

$$\iiint_R \frac{4}{65} z \, dV$$