# University of Notre Dame Calculus III

LECTURE 22:TRIPLE INTEGRALS USING SPHERICAL COORDINATES

## Spherical Coordinates

As we can use cylinders to give coordinates on $\mathbb{R}^3$ we can also use spheres. These coordinates are obtained by rotating polar coordinates into $\mathbb{R}^3$. Spherical coordinates are $(\rho,\theta,\phi)$ where $\rho$ is the distance from the origin, $\theta$ is the angle made with the positive $x$-axis in the $xy$-plane, and $\phi$ is the angle made with the positive $z$-axis. So, we have

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi \, .$$

The relation to cartesian is

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi$$

We also have

$$\rho^2 = x^2 + y^2 + z^2$$

**Example 1.**

a) Write the point with spherical coordinates $\left(3, \frac{\pi}{2}, \frac{3\pi}{4}\right)$ in cartesian coordinates.

b) Write the point with cartesian coordinates $(-1, 1, -\sqrt{2})$ in spherical coordinates.

**Solution:**

In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

**Example 2.** Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4z$ and above the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$.

**Solution:**

**Extra Problems**

1. Describe the surfaces whose equations in cylindrical coordinates are $(a)\ \rho = c$, $(b)\ \theta = c$, $(c)\ \phi = c$.
2. Suppose $R$ is the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, under the cone $z = \sqrt{x^2 + y^2}$, and above the cone $z = -\sqrt{x^2 + y^2}$. Write the following triple integral in spherical coordinates
$$\iiint_R z^2 \, dV.$$
3. Suppose $E$ is the region bounded by the spheres $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 9$ and above the cone $\phi = \pi/3$. Evaluate
$$\iiint_R \frac{4}{65} z \, dV$$