University of Notre Dame Calculus III

LECTURE 23: APPLICATIONS WITH MASS

Mass and Center of Mass

Recall that the center of mass of a system is the point, if all the mass of the system were concentrated there, the total moment is the same as the original.

Discrete Case

In the discrete case of masses m_i at points (x_i, y_i), the total moment or moment of mass about the

1. y-axis is $M_y = \sum_i m_i x_i$

2. x-axis is $M_x = \sum_i m_i y_i$

So, the center of mass has coordinates $(\overline{x}, \overline{y})$ where

$$\left(\sum_i m_i\right)\overline{x} = \sum_i m_i x_i \qquad \left(\sum_i m_i\right)\overline{y} = \sum_i m_i y_i$$

Thus

$$\overline{x} = \frac{M_y}{\text{total mass}} \qquad \overline{y} = \frac{M_x}{\text{total mass}}$$

We can do this for a system in $\mathbb{R}^3$ as well,

$$\overline{z} = \frac{M_{xy}}{\text{total mass}}$$

where $M_{xy} = \sum_i m_i z_i$ is the total moment about the xy-plane.

Continuous Case

Let's suppose we had a lamina D in $\mathbb{R}^2$ with density function $\rho(x, y)$. Then the total mass of D is

$$\boxed{\text{mass} = m = \int\int_D \rho(x, y)\, dA}$$

Using the discrete case as a guide, we find that

$$M_y = \int\int_D x\rho(x, y)\, dA \qquad M_x = \int\int_D y\rho(x, y)\, dA$$

So the center of mass of D has coordinates

$$\boxed{(\overline{x}, \overline{y}) = \left(\frac{\int\int_D x\rho\, dA}{\int\int \rho\, dA}, \frac{\int\int_D y\rho\, dA}{\int\int_D \rho\, dA}\right)}$$

The equations are similar for solid regions in $\mathbb{R}^3$.

Extra Problems

1. Consider a lamina in the region D bounded by $y = 1 - x^2$ and $y = 0$ with density $\rho(x, y) = ky$ for some constant $k > 0$. Find the total mass of the lamina.

2. Find the center of mass of the lamina of the previous problem.

3. Can you conceive of a way to extend the concept of center of mass to three dimensions?