

University of Notre Dame Calculus III

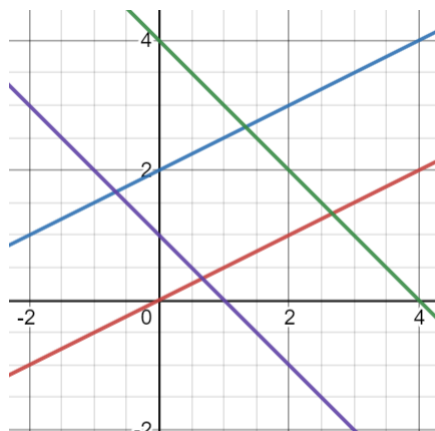
LECTURE 24: CHANGE OF VARIABLES

Change of Variables

Example 1. Compute $\iint_D 3xy \, dA$ where D is the region bounded by $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.

Solution:

Sketch:



Notice that, to do this integral would require splitting the region into 3 pieces. There must be an easier way... Notice that the opposite sides of the parallelogram are described by the same function

$$x - 2y = -4, 0 \quad x + y = 1, 4$$

If we write $u = x - 2y$ and $v = x + y$, then the region is bounded by $u = -4$, $u = 0$, $v = 1$, $v = 4$ in the uv -plane... much simpler! We need to replace x and y , so we solve for them in terms of u and v :

$$\begin{cases} u = x - 2y \\ v = x + y \end{cases}$$

Using algebra we solve for x and y giving us $x = \frac{1}{3}(2v + u)$ and $y = \frac{1}{3}(v - u)$. Our integrand becomes

$$3xy = \frac{1}{3}(v - u)(2v + u) = \frac{1}{3}(2v^2 - uv - u^2)$$

What about dA ? Just like when we learned u -substitution in Calc 1, we need to find a replacement something else to switch from the xy -coordinate system to the uv -coordinate system. This is called the Jacobian, which is expressed like this:

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

this $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the transformation and is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

So in this example

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} - \left(-\frac{2}{9}\right) = \frac{1}{3}$$

thus $dA = \frac{1}{3} du dv$. So, the integral becomes

$$\iint_D 3xy dA = \int_1^4 \int_{-4}^0 \frac{1}{9} (2v^2 - uv - u^2) du dv = \frac{164}{9}$$

If we write $T(u, v) = (x(u, v), y(u, v))$ to represent the transformation, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \det DT(u, v)$$

Definition 1. $T(u, v) = (x(u, v), y(u, v))$ is C^1 if its components have continuous first partials.

Change of Variables Formula (2 variables)

Suppose $T(u, v) = (x(u, v), y(u, v))$ is C^1 and sends the region S in the uv -plane to the region R in the xy -plane. If the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ is nonzero at all points in S , $f(x, y)$ is continuous on R , and T is one-to-one on S , except maybe on the boundary of S , then

$$\begin{aligned}\iint_R f(x, y) dA &= \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\ &= \iint_{T^{-1}(R)} f(T(u, v)) |\det DT| du dv\end{aligned}$$

This theorem can also be used to make integrands simpler. This more like u -substitution. Before an example of this, a neat trick from linear algebra:

$$\det(A^{-1}) = \frac{1}{\det A}$$

If $T(u, v) = (x(u, v), y(u, v))$, then $T^{-1}(x, y) = (u(x, y), v(x, y))$. So, $\frac{\partial(u, v)}{\partial(x, y)} = \det D(T^{-1})$. But $D(T^{-1}) = DT^{-1}$. Thus

$$\frac{\partial(u, v)}{\partial(x, y)} = \det D(T^{-1}) = (\det DT)^{-1} = \frac{1}{\det DT} = \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}$$

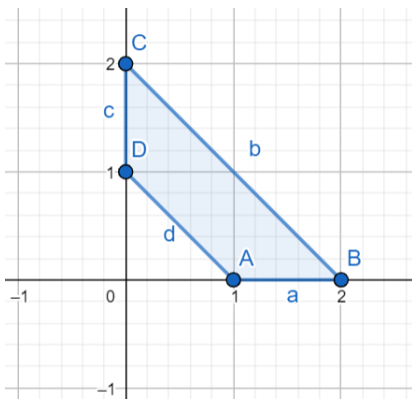
So

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

Example 2. Compute $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, $(0, 1)$.

Solution:

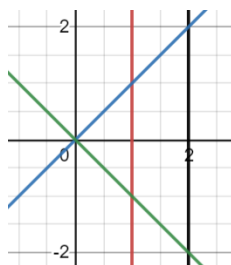
$\cos\left(\frac{y-x}{y+x}\right)$ looks pretty hard to integrate... If we write $u = y - x$ and $v = y + x$, then we get $\cos\left(\frac{y-x}{y+x}\right) = \cos\left(\frac{u}{v}\right)$, a bit better. What happens to R ? More like, what maps to R ? First, let's sketch R :



What is the region S which maps to R ?

xy -plane	uv -plane
$x + y = 2$	$v = 2$
$x + y = 1$	$v = 1$
$x = 0$	$\begin{cases} u = y - x = y \\ v = y + x = y \end{cases} \implies u = v$
$y = 0$	$\begin{cases} u = y - x = -x \\ v = y + x = x \end{cases} \implies u = -v$

So, S is bounded by $v = 2$, $v = 1$, $u = v$, and $u = -v$. Pictorially:



Now, we need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

Thus,

$$\begin{aligned} \iint_R \cos\left(\frac{y-x}{y+x}\right) dA &= \int_1^2 \int_{-v}^v \cos\left(\frac{u}{v}\right) \left| \frac{-1}{2} \right| du dv \\ &= \frac{1}{2} \int_1^2 \int_{-v}^v \cos\left(\frac{u}{v}\right) du dv = \frac{3}{2} \sin(1) \end{aligned}$$

There is a corresponding 3-variable version of the theorem. If $T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ and $T(S) = R$ then

$$\int \int \int_R f(x, y, z) dV = \int \int \int_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Extra Problems

1. Let $x = \frac{u}{v}$ and $y = uv$. Compute the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$.
2. Evaluate $\iint_D xy dx dy$, where D is the region in the first quadrant bounded by curves $xy = 2$, $xy = 4$, $y = x$, and $y = 2x$. (Hint: consider $u = xy$ and $v = y/x$ and use the change of variables obtained by expressing x and y in terms of u and v .)
3. Suppose R is the parallelogram in the xy -plane with vertices $(0, 0), (2, 1), (3, 3), (1, 2)$. Use the change of variables $x = u + 2v$, $y = 2u + v$ to compute the integral

$$\iint_R (2x - y)^2 dA.$$

4. Use the transformation $x = u^2$ and $y = v^2$ to find the area of the region bounded by the curves $\sqrt{x} + \sqrt{y} = 1$, x -axis and y -axis.