University of Notre Dame Calculus III

Change of Variables

Example 1. Compute $\int\int_D 3xy\,dA$ where D is the region bounde by $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.

Solution:

If we write $T(u,v)=(x(u,v),y(u,v))$ to represent the transformation, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \det DT(u,v)$$

Definition 1. *$T(u,v)=(x(u,v),y(u,v))$ is C^1 if its components have continuous first partials.*

Change of Variables Formula (2 variables)

Suppose $T(u,v) = (x(u,v), y(u,v))$ is C^1 and sends the region S in the uv-plane to the region R in the xy-plane. If the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero at all points in S, $f(x,y)$ is continuous on R, and T is one-to-one on S, except maybe on the boundary of S, then

$$\int\int_R f(x,y)\,dA = \int\int_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
$$= \int\int_{T^{-1}(R)} f(T(u,v))|\det DT|\,du\,dv$$

This theorem can also be used to make integrands simpler. This more like u-substitution. Before an example of this, a neat trick from linear algebra:

$$\det(A^{-1}) = \frac{1}{\det A}$$

If $T(u,v) = (x(u,v), y(u,v))$, then $T^{-1}(x,y) = (u(x,y), v(x,y))$. So, $\frac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1})$. But $D(T^{-1}) = DT^{-1}$. Thus

$$\frac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1}) = (\det DT)^{-1} = \frac{1}{\det DT} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

So

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

Example 2. Compute $\int\int_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

Solution:

There is a corresponding 3-variable version of the theorem. If $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ and $T(S) = R$ then

$$\int\int\int_R f(x,y,z)\,dV = \int\int\int_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Extra Problems

1. Let $x = \dfrac{u}{v}$ and $y = uv$. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

2. Evaluate $\iint_D xy\,dx\,dy$, where D is the region in the first quadrant bounded by curves $xy = 2$, $xy = 4$, $y = x$, and $y = 2x$. (Hint: consider $u = xy$ and $v = y/x$ and use the change of variables obtained by expressing x and y in terms of u and v.)

3. Suppose R is the parallelogram in the xy-plane with vertices $(0,0), (2,1), (3,3), (1,2)$. Use the change of variables $x = u+2v$, $y = 2u+v$ to compute the integral

$$\iint_R (2x-y)^2\,dA.$$

4. Use the transformation $x = u^2$ and $y = v^2$ to find the area of the region bounded by the curves $\sqrt{x}+\sqrt{y} = 1$, x-axis and y-axis.