University of Notre Dame Calculus III

LECTURE 25:

Vector Fields

Definition 1. *A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ which assigns to each point (x, y) in its domain a 2-D vector $\vec{F}(x, y)$. We often write $\vec{F}$ in terms of its component functions:*

$$\begin{aligned}\vec{F}(x,y) &= \langle P(x,y), Q(x,y)\rangle = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} \\ &= \langle P, Q\rangle = P\hat{i} + Q\hat{j}\end{aligned}$$

Of course there are vector fields on $\mathbb{R}^3$ as well:

$$\vec{F}(x,y,z) = \langle P, Q, R\rangle = P\hat{i} + Q\hat{j} + R\hat{k}$$

Let's sketch some vector fields:
Example 1. Sketch the vector field:

$$\vec{F}(x,y) = \langle -1, 1\rangle \quad \vec{F}(x,y) = \langle x, y\rangle \quad \vec{F}(x,y) = \langle 2y, 0\rangle$$

Solution:

An example from physics:
Suppose there is a body of mass M at $(0,0,0)$. The gravitational force exerted on a body of mass m with position vector $\vec{R} = \langle r\rangle = \langle x, y, z\rangle$ is

$$\vec{F}(x,y,z) = \vec{R}(\vec{r}) = -\frac{mMG}{||\vec{r}||^2}\frac{\vec{r}}{||\vec{r}||} = \frac{-mMG}{||\vec{r}||^3}\vec{r}$$

Gradient Vector Fields

Definition 2. *A gradient vector field is a vector field of the form ∇f for some function $f(x,y)$ or $f(x,y,z)$. So*

$$\nabla f(x,y) = \langle f_x, f_y \rangle \qquad \nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

Example 2. $\vec{F} = \langle -1, 1 \rangle$ is the gradient vector field where $f = y - x$

Example 3. $\vec{F} = \langle x, y, z \rangle$ is the gradient vector field of $f = \frac{1}{2}(x^2 + y^2 + z^2)$

We call a vector field $\vec{F}$ conservative if there exists some scalar function f such that $\vec{F} = \nabla f$. The scalar function, f, is called the potential function for $\vec{F}$.

Extra Problems

1. Plot the vector field $\mathbf{F}(x,y) = \langle -y, x \rangle$.
2. Find the gradient vector field of $f(x,y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?
3. Find a potential function for the vector field

$$F = (2x\ln(y) + 3x^2)\, i + \left(\frac{x^2}{y} + \cos(y)\right) j.$$

Line Integrals

Suppose we have a curve C, parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, in $\mathbb{R}^2$ which is smooth. Let $f(x,y)$ be a function whose domain includes C. We could ask about integrating f along C... how would we do this? Same idea as usual: cut C into pieces and take sample points and create a Riemann sum. We do this using $\vec{r}$:

Cut up $[a,b]$ into n pieces $[t_{i-1}, t_i]$ and let $P_j = \vec{r}(t_j)$, $0 \le j \le n$. This cuts C into n pieces of arclength ΔS_i. Inside each $[t_{i-1}, t_i]$, choose a sample t_i^*. This gives us sample points along $C : (x_i^*, y_i^*) = \vec{r}(t_i^*)$. Thus we define

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Recall the arc length function

$$s(t) = \int_a^t ||\vec{r}\,'(u)||\,du$$

So

$$\frac{ds}{dt} = ||\vec{r}\,'(t)|| \Longrightarrow ds = ||\vec{r}\,'(t)||\,dt$$

The effective way to compute $\int_C f(x,y)\,ds$ is

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,||\vec{r}\,'(t)||\,dt}$$

(Notice there is nothing stopping us from having a curve C in $\mathbb{R}^3$ and a function $f = f(x,y,z)$. The right side of the formula will be exactly the same.) The integral $\int_c f\,ds$ is called a scalar line integral or an integral with respect to arclength or a line integral of f along C.

Example 4. Compute the line integral of $f(x,y) = 18y^3$ along the piece of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$.

Solution:

If C is a piecewise-smooth curve,i.e., a collection $C_1, ..., C_n$ of smooth curves joined end on end. Then,

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds$$

Application Suppose we have a thin wire bent in the shape of a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) with linear density function ρ. Then the mass of the wire is $m = \int_C \rho\,ds$ and the center of mass has coordinates:

$$\overline{x} = \frac{1}{m}\int_C x\rho\,ds \qquad \overline{y} = \frac{1}{m}\int_C y\rho\,ds \qquad \overline{z} = \frac{1}{m}\int_C z\rho\,ds$$

In the definition of $\int_C f\,ds$, replacing Δs, by simply Δx_i or Δy_i yields two other types of line integrals: the line integral of f along with respect to

- x:

$$\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))x'(t)\,dt$$

- y

$$\int_C f(x,y)\,dy = \int_a^b f(\vec{r}(t))y'(t)\,dt$$

where C is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$.
There's even the case where these happen together

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P\,dx + \int_C Q\,dy$$

Example 5. Compute the line integral $\int_C y\,dx - x\,dy + z\,dz$ where C is the path given by
a) C_1: $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$
b) C_2: line segment from $(1,0,0)$ to $(1,0,2\pi)$
c) C_3: line segment from $(1,0,2\pi)$ to $(1,0,0)$

Solution:

Notice how a) and b) started and ended at the same spot, but yielded different answers. This means these types of integrals depend on the path of integration. Notice how the path in b) and c) is the same, just traveled in opposite directions. This implies dependence on the orientation of the path of integration (direction of travel) as well. In particular, the answers are negatives of each other. If C has an orientation, $-C$ denotes the same curve with opposite orientation. We have

$$\int_{-C} f\,dx = -\int_C f\,dx$$
$$\int_{-C} f\,dy = -\int_C f\,dy$$
$$\int_{-C} f\,dz = -\int_C f\,dz$$

However, since arclength doesn't depend on orientation,

$$\int_{-C} f\,ds = \int_C f\,ds$$

Consider again the line integral

$$\int_C P\,dx + Q\,dy + R\,dz$$

We can rewrite it as

$$\begin{aligned}\int_C P\,dx + Q\,dy + R\,dz &= \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle \\ &= \int_C \vec{F} \cdot d\vec{r}\end{aligned}$$

Where $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ is a vector field and $\vec{r} = \langle x, y, z\rangle$. These integrals are called vector line integrals or line integrals of $\vec{F}$ along C. Their main purpose is to compute work.

Let's suppose we have a particle moving along a path C in the presence of a force field $\vec{F}$. For constant forces $\vec{F}$ the work performed in moving in a straight line is $W = \vec{F} \cdot \vec{d}$, where $\vec{d}$ is the displacement vector. In the (usual) case when C is not a straight line, we approximate the work done over small segments of C, then add them up.

The work performed over the segment from P_{i-1} to P_i is

$$W_i \approx \vec{F}(P_i^*) \cdot \vec{T}(t_i^*)\Delta s_i$$

giving, in the limit

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \vec{T}(t_i^*)\Delta s_i = \int_C \vec{F} \cdot \vec{T}\,ds$$

Using known facts:

$$\begin{aligned}W = \int_C \vec{F} \cdot \vec{T}\,ds &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \left(\frac{\vec{r}\,'(t)}{||\vec{r}(t)||}\right) ||\vec{r}(t)||\,dt \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot d\vec{r}\end{aligned}$$

Example 6. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, 3y^2\rangle$ and C is parametrized by $\vec{r}(t) = \langle 11t^4, t^3\rangle$, $0 \le t \le 1$.

Solution:

Example 7. Find the work done by $\vec{F} = \langle e^z, xz, x + y\rangle$ in moving a particle from the origin to $(1, 1, -1)$ along $\vec{r}(t) = \langle t^2, t^3, -t\rangle$.

Solution: