University of Notre Dame Calculus III

LECTURE 26: THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

Throughout the rest of the class we will be generalizing the result so fundamental, it says so in its name: the fundamental theorem of calculus

$$\int_a^b \frac{d}{dx} f(x)\, dx = f(b) - f(a)$$

There will be a theme to each of these we will be trading derivatives for boundaries (or vice versa). Notice $\partial[a,b] = \{a,b\}$.

First we need some terminology.

Conservative Vector Fields and Potentials

If $\vec{F} = \langle P, Q\rangle$ is conservative, then $P = f_x$ and $Q = f_y$ for some $f = f(x,y)$. So, by Clairaut's theorem $P_y = Q_x$ (as long as $\vec{F}$ is C^1). This gives us a test for conservative vector fields. Let $\vec{F}$ be a C^1 vector field.
(a) If $\vec{F}(x,y) = \langle P, Q\rangle$ and $\vec{F}$ is conservative, then

$$P_y = Q_x$$

(b) If $\vec{F}(x,y,z) = \langle P, Q, R\rangle$ and $\vec{F}$ is conservative, then

$$\operatorname{curl}\vec{F} = \vec{0}$$

Note: We don't know what curl is yet, but will learn shortly.

These are tests we can use to determine whether $\vec{F}$ is not conservative. If $P_y \neq Q_x$ in (a) or $\operatorname{curl}\vec{F} = \vec{0}$ in (b), then $\vec{F}$ is not conservative. If the equations hold then we don't know if $\vec{F}$ is conservative or not. A curiosity would be to know when these results are reversible, i.e., when $P_y = Q_x$ or $\operatorname{curl}\vec{F} = \vec{0}$ implies $\vec{F}$ is conservative. To answer this, we need two definitions.

Definition 1. *A curve which does not cross itself anywhere between its endpoints is called simple.*

Definition 2. *A region D is called simply connected if D is connected and every simple closed curve in D encloses only points in D.*

Here are the reversals:

Theorem 3. *a) If $\vec{F} = \langle P, Q\rangle$ is a C^1 vector field on an open, simply connected region D, and $P_y = Q_x$, we have that $\vec{F}$ is conservative.*
b) If $\vec{F} = \langle P, Q, R\rangle$ is a C^1 vector field on all of $\mathbb{R}^3$ and $\operatorname{curl}\vec{F} = \vec{0}$, then $\vec{R}$ is conservative.

Often, the easiest way to determine whether a vector field is conservative is to just try to find a potential.

The Fundamental Theorem of Line Integrals

Theorem 4. *Fundamental Theorem of Line Integrals Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of 2 or 3 variables whose gradient is continuous on C. Then*

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla d f(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a))\end{aligned}$$

□

Definition 5. *We say a line integral $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if given any two curves C_1 and C_2 in the domain of $\vec{F}$ which start and end at the same place we have*

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

The theorem tells us then that line integrals of conservative vector fields are independent of path.

Definition 6. *A curve is <u>closed</u> if it starts and ends at the same point.*

Theorem 7. *$\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C in D.*

Definition 8. *A set D is <u>open</u> if for every point P in D, we can fit a disk of radius $\epsilon > 0$ (ϵ as small as needed) around P inside D.*

Definition 9. *A set D is <u>connected</u> if any two points in D can be joined by a path in D.*

Theorem 10. *Suppose $\vec{F}$ is a vector field which is continuous on an open and connected set D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative on D.*

Example 1. Compute $\int_C \left(\ln y + 2xy^3\right) dx + \left(3x^2y^2 + \frac{x}{y}\right) dy$ where C has parametric equations

$$x = \frac{1}{2}t^2 + 2, \qquad y = e^t(1 + 2t - t^2), \qquad 0 \le t \le 2.$$

Solution:

Both the curve and the integrand look difficult so, let's try to use the FToLI: First, we need a potential for:

$$\vec{F} = \langle \ln y + 2xy^3, 3x^2y^2 + \frac{x}{y} \rangle = \langle P, Q \rangle$$

If $\vec{F}$ is conservative $\vec{F} = \langle f_x, f_y \rangle = \langle P, Q \rangle$, so

$$f = \int P\, dx = \int (\ln y + 2xy^3)\, dx = x \ln y + x^2y^3 + g(y)$$

$$f_y = \frac{x}{y} + 3x^2y^2 + g'(y) = Q = 3x^2y^2 + \frac{x}{y} \Longrightarrow g'(y) = 0 \Longrightarrow g(y) = K$$

Implies

$$f = x \ln y + x^2y^3 + K$$

We can chose any value for K that we want since we just need a potential. So, let's take $K = 0$. The endpoints of C are $(2,1)$ and $(4, e^2)$, so

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(4, e^2) - f(2,1) = (4 \ln e^2 + 16e^6) - (2 \ln 1 + 4) \\ &= 8 + 16e^6 - 4 = 4 + 16e^6\end{aligned}$$

Let's do an application to end this section: Let $\vec{F}$ be a continuous force field which moves an object along a path C given by $\vec{r}(t)$, $a \le t \le b$ where $\vec{r}(a) = A$ and $\vec{r}(b) = B$. By Newton's second law

$$\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$$

Thus, the work done by $\vec{F}$ is then

$$\begin{aligned}W = \int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{d}F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b m\vec{r}''(t) \cdot \vec{r}'(t)\, dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt}[\vec{r}'(t) \cdot \vec{r}'(t)]\, dt = \frac{m}{2} \int_a^b \frac{d}{dt} ||\vec{r}'(t)||^2\, dt \\ &= \frac{1}{2} m ||\vec{r}'(t)||^2 \Big|_a^b = \frac{1}{2} m |\vec{r}'(b)||^2 - \frac{1}{2} m ||\vec{r}'(a)||^2 \\ &= \frac{1}{2} m \left(||\vec{v}(b)||^2 - ||\vec{v}(a)||^2 \right)\end{aligned}$$

The quantity $\frac{1}{2} m (||\vec{v}(t)||)^2$ is the kinetic energy at time t, so we can write $W = K(B) - K(A)$. Now, let's assume that the force $\vec{F}$ is conservative. Then $\vec{F} = \nabla f$. Typically, in Physics, the potential energy is defined as $P = -f$ so that $\vec{F} = -\nabla P$. Then, the FToLI gives

$$W = \int_C \vec{F} \cdot d\vec{r} = -\int_A^B \nabla P \cdot d\vec{r} = -(P(B) - P(A)) = P(A) - P(B)$$

Combining this with what we know about kinetic energy we have

$$K(B) - K(A) = W = P(A) - P(B) \qquad \Longleftrightarrow \qquad P(A) + K(A) = P(B) + K(B)$$

This is the Law of Conservation of Energy.
Let's do an example of finding a potential for a 3D vector field.
Example 2. Determine whether the vector field

$$\vec{F} = \langle e^x \sin yz, ze^x \cos yz, ye^x \cos yz + 3z^2 \rangle$$

is conservative. If so, find a potential.

Then find

$$\int_C \vec{F} \cdot dr$$

where C is a smooth curve where given by $\vec{r}(t)$ that starts at $\vec{r}(0) = \langle 0, \pi/2, 1 \rangle$ and ends at $\vec{r}(\pi) = \langle 1, 0, 2 \rangle$.

Solution:

Let's just try to find one. If $\vec{F}$ is conservative

$$\vec{F} = \langle P, Q, R \rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

So,

$$f = \int P\, dx = e^x \sin yz + g(y, z)$$

Since $f_y = ze^x \cos yz + g_y(y, z) = Q = ze^x \cos yz$ we know $g_y(y, z) = 0$ so $g(y, z) = h(z)$.
Since $f_z = ye^x \cos yz + h'(z) = R = ye^x \cos yz + 3z^2$ $h'(z) = 3z^2$ implies $z^3 + K$.
Thus

$$f = e^x \sin yz + z^3$$

is a potential.

Thus

$$\int_C \vec{F} \cdot dr = f(1, 0, 2) - f(0, \pi/2, 1) = 8 - (1 + 1) = 6.$$