

University of Notre Dame Calculus III

LECTURE 27: GREEN'S THEOREM

Green's Theorem

Definition 1. A curve C (in the plane) has positive orientation if it is traversed counterclockwise, exactly once. If C is the boundary of a region D in the plane, C has positive orientation if when you traverse C , D is on the left. We write $C = \partial D$ in this situation.

Theorem 2. *Green's Theorem* Let C be a positively oriented piecewise smooth, simple closed curve in the plane which bounds a region D . If P and Q have continuous first partials on a region containing D . Then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notation: For a closed curve C , we write

$$\oint_C P dx + Q dy$$

to imply that C has positive orientation. Recall that writing ∂D for the boundary of D implies ∂D has positive orientation. We could then rewrite Green's Theorem as

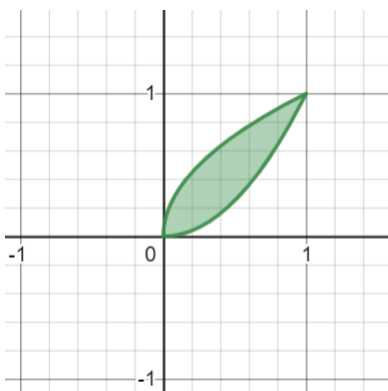
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P dx + Q dy.$$

Example 1.

Compute $\oint_C (y + e^x) dx + (2x + \cos y^2) dy$ where C is the boundary of the region bounded by $y = x^2$ and $x = y^2$.

Solution:

The region is



This is impossible to do directly. So let's use Green's Theorem on it:

$$\begin{aligned} \oint_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy &= \iint_D [(2) - (1)] dA = \iint_D dA \\ &= \int_0^1 \int_{y^2}^{\sqrt{y}} dx dy = \int_0^1 (\sqrt{y} - y^2) dy = \left(\frac{2}{3} y^{\frac{3}{2}} - \frac{1}{3} y^3 \right) \Big|_0^1 = \frac{1}{3} \end{aligned}$$

A common problem in double integration is finding the area of a region. We can use Green's theorem to translate this into a line integral around the boundary of the region

$$\text{Area}(D) = \iint_D 1 dA = \oint_{\partial D} x dy = - \oint_{\partial D} y dx = \frac{1}{2} \oint_{\partial D} x dy - y dx$$

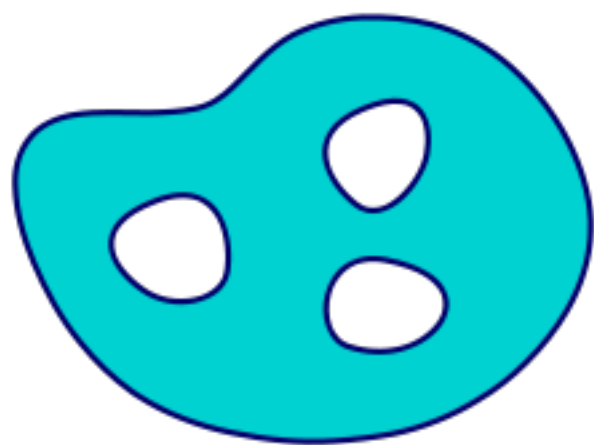
Sometimes we can come across regions with holes in them (e.g. annuli). A fair question to ask would be if Green's theorem can deal with them. Let's begin with the following: If $D = D_1 \cup D_2$ then

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$$

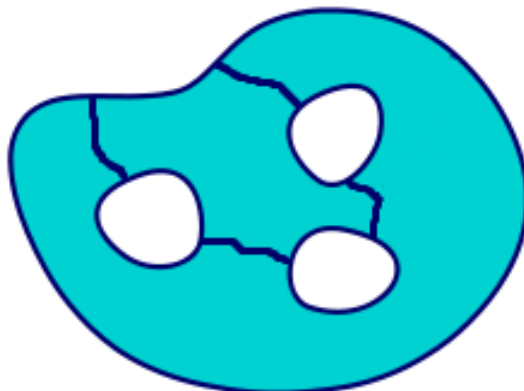
Then, Green's theorem tells us:

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D_1} (Q_x - P_y) dA + \iint_{D_2} (Q_x - P_y) dA \\ &= \oint_{\partial D_1} P dx + Q dy + \oint_{\partial D_2} P dx + Q dy \\ &= \left(\int_{C_1} P dx + Q dy + \int_{C_3} P dx + Q dy \right) + \left(\int_{C_2} P dx + Q dy - \int_{C_3} P dx + Q dy \right) \\ &= \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy \\ &= \oint_{\partial D} P dx + Q dy \end{aligned}$$

The purpose of this was to show that cutting regions doesn't change the integral. Now, if D has holes:



Notice that the orientation of ∂D gives C_1 the counterclockwise orientation and C_2 the clockwise orientation. Now, we can cut D into 2 regions without holes:



So, this gives:

$$\begin{aligned}
 \iint_D (Q_x - P_y) dA &= \iint_{D_1} (Q_x - P_y) dA + \iint_{D_2} (Q_x - P_y) dA \\
 &= \left(\int_{C_1} P dx + Q dy + \int_{C_3} P dx + Q dy + \int_{C_2'} P dx + Q dy + \int_{C_4} P dx + Q dy \right) \\
 &\quad + \left(\int_{C_1''} P dx + Q dy + \int_{-C_4} P dx + Q dy + \int_{C_2''} P dx + Q dy + \int_{-C_3} P dx + Q dy \right) \\
 &= \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy \\
 &= \oint_{C_1} P dx + Q dy - \oint_{C_2} P dx + Q dy
 \end{aligned}$$

So, when there's holes, we integrate around the outside, then subtract off the integrals around the holes.

Extra Problems

1. Evaluate $\int_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$ where C is the circle $x^2 + y^2 = 9$ oriented in counterclockwise fashion.
2. Evaluate $\int_C y^2 dx + 3xy dy$ where C is the positively oriented boundary of the region in the upper-half-plane trapped between $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.