University of Notre Dame Calculus III

Green's Theorem

Definition 1. *A curve C (in the plane) has positive orientation if it is traversed counterclockwise, exactly once. If C is the boundary of a region D in the plane, C has positive orientation if when you traverse C, D is on the left. We write $C = \partial D$ in this situation.*

Theorem 2. *Green's Theorem Let C be a positively oriented piecewise smooth, simple closed curve in the plane which bounds a region D. If P and Q have continuous first partials on a region containing D. Then*

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Notation: For a closed curve C, we write

$$\oint_C P\,dx + Q\,dy$$

to imply that$C$ has positive orientation. Recall that writing ∂D for the boundary of D implies ∂D has positive orientation. We could then rewrite Green's Theorem as

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy\,.$$

Example 1.

Compute $\oint_C \left(y + e^x\right) dx + \left(2x + \cos y^2\right) dy$ where C is the boundary of the region bounded by $y = x^2$ and $x = y^2$.

Solution:

A common problem in double integration is finding the area of a region. We can use Green's theorem to translate this into a line integral around the boundary of he region

$$\text{Area}(D) = \int\int_D 1\,dA = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$$

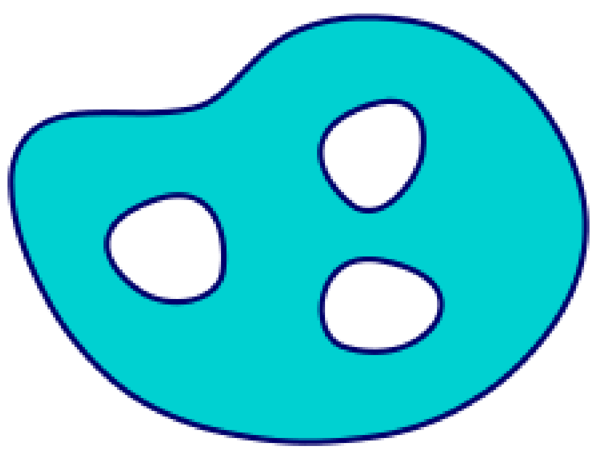
Sometimes we can come across regions with holes in them (e.g. annuli). A fair question to ask would be if Green's theorem can deal with them. Let's begin with the following: If $D = D_1 \cup D_2$ then

$$\int\int_D f\,dA = \int\int_{D_1} f\,dA + \int\int_{D_2} f\,dA$$

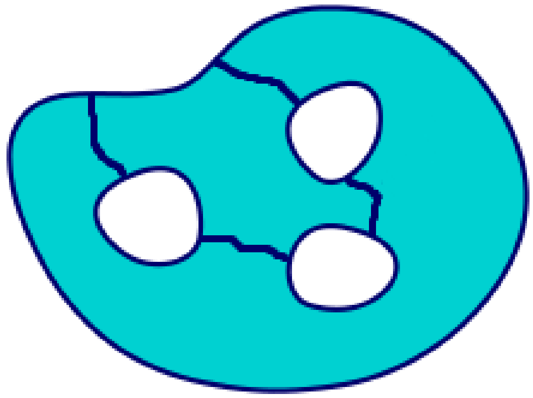
Then, Green's theorem tells us:

$$\begin{aligned}
\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \int\int_{D_1} (Q_x - P_y)\,dA + \int\int_{D_2} (Q_x - P_y)\,dA \\
&= \oint_{\partial D_1} P\,dx + Q\,dy + \oint_{\partial D_2} P\,dx + Q\,dy \\
&= \left(\int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy\right) + \left(\int_{C_2} P\,dx + Q\,dy - \int_{C_3} P\,dx + Q\,dy\right) \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&= \oint_{\partial D} P\,dx + Q\,dy
\end{aligned}$$

The purpose of this was to show that cutting regions doesn't change the integral. Now, if D has holes:



Notice that the orientation of ∂D gives C_1 the counterclockwise orientation and C_2 the clockwise orientation. Now, we can cut D into 2 regions without holes:



So, this gives:

$$\begin{aligned}
\int\int_D (Q_x - P_y)\, dA &= \int\int_{D_1} (Q_x - P_y)\, dA + \int\int (Q_x - P_y)\, dA \\
&= \left(\int_{C_1'} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_2'} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy\right) \\
&\quad + \left(\int_{C_1''} P\,dx + Q\,dy + \int_{-C_4} P\,dx + Q\,dy + \int_{C_2''} P\,dx + Q\,dy + \int_{-C_3} P\,dx + Q\,dy\right) \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&= \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy
\end{aligned}$$

So, when there's holes, we integrate around the outside, then subtract off the integrals around the holes.

Extra Problems

1. Evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4 + 1})\,dy$ where C is the circle $x^2 + y^2 = 9$ oriented in counterclockwise fashion.

2. Evaluate $\int_C y^2\,dx + 3xy\,dy$ where C is the positively oriented boundary of the region in the upper-half-plane trapped between $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.