University of Notre Dame Calculus III

LECTURE 28: CURL AND DIVERGENCE

Curl and Divergence

Definition 1. *The operator* ∇ *(pronounced "del") is*

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

We've already seen one use

$$\nabla f = grad f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Definition 2. *Let* $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ *be a vector field*

- *the divergence of* $\vec{F}$ *is*

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- *the curl of* $\vec{F}$ *is*

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F}$$
$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Notice that to find the curl $\vec{F}$ must be a 3D vector field!

Example 1. Find the curl and divergence of

$$\vec{F} = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$$

Solution:

$$\operatorname{curl} \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$
$$= \langle 3x^2y^2z - 2x^3yz, 3xy^2z^2 - 2xy^3z, 3x^2yz^2 - 2xyz^3 \rangle$$
$$\operatorname{div}(\vec{F}) = y^2z^3 + x^3z^2 + x^2y^3$$

We have the useful facts:

Theorem 3. *Let* $\vec{F} = \langle P, Q, R \rangle$ *and* $f = f(x,y,z)$ *and suppose* P, Q, R, *and* f *are* C^2. *Then*

- $\operatorname{curl}(\operatorname{grad} f) = \vec{0} = \operatorname{curl}(\nabla f)$
- $\operatorname{div}(\operatorname{curl} \vec{F}) = \vec{0}$

The proof of each of these follow from Clairaut's theorem. Notice that the first point says that the curl of a conservative vector field is $\vec{0}$.

Extra Problems

1. Compute the curl and divergence of fields $F = \langle x, y, 0\rangle$ and $F = \langle -y, x, 0\rangle$.
2. Compute divergence and curl of $F = \langle x, y, z\rangle$ and $F = \langle x, -z, y\rangle$.