University of Notre Dame Calculus III

LECTURE 29 AND 30:PARAMETRIC SURFACE

Parametric Surfaces

Suppose we have a surface S in $\mathbb{R}^3$. Since a surface is inherently 2-dimensional, it requires 2 variables to parametrize. A parametrization of S looks like $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ where D is the domain.

Example 1. Identify and sketch the surface parrametrized by:

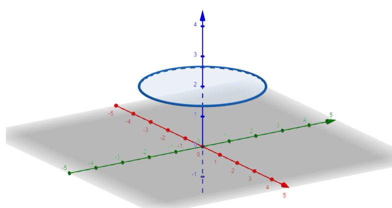
a) $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2\rangle$, $0 \le u \le 1$, $0 \le v \le 2\pi$

b) $\vec{r}(s,t) = \langle s, \pi, t\rangle$, $s, t \in \mathbb{R}$

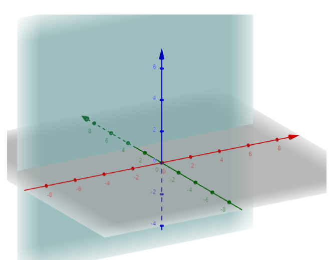
c) $\vec{r}(u,v) = \langle u, v, u^2 + v^2\rangle$, $u, v \in \mathbb{R}$

Solution:

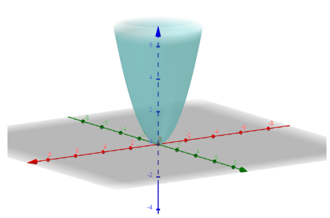
a) Notice that $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2$. So, since u goes from 0 to 1, we get a whole disk, and $z = 2$ means it's at height 2



b) Here x and z can be anything, while y is stuck at π. So, the graph is the plane $y = \pi$:



c) The components satisfy $x^2 + y^2 = z$, so the surface is a paraboloid:



Recall that the paraboloid is described in cylindrical coordinates $z = r^2$, so a reparametrization of

(c) is

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle, \qquad r \ge 0, 0 \le \theta \le 2\pi$$

Suppose we have a surface given by $z = f(x, y)$, $(x, y) \in D$.
Since z is completely determined by x and y, we can let x and y be the parameters, then

$$\vec{r}(x, y) = \langle x, y, f(x, y)\rangle, \qquad (x, y) \in D$$

is a parametrization of the surface.
Example 2. Parametrize the surface $z = 3\sqrt{x^2 + y^2}$.

Solution:

Using the above, this is parametrized by

$$\vec{r}(x, y) = \langle x, y, 3\sqrt{x^2 + y^2}\rangle$$

However, to give it nicer grid curves, it's better to use cylindrical coordinates. So $z = 3r$ and

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle, \qquad 0 \le r,\ 0 \le \theta \le 2\pi$$

Example 3. Parametrize the sphere $x^2 + y^2 + z^2 = 81$.

Solution:

Since we cannot write down the sphere as the graph of a function, we have to try something else. Luckily, we have a coordinate system which nicely describes spheres: spherical coordinates. This sphere is given by $\rho = 9$. Using the equations from spherical coordinates we find parametric equations for the sphere

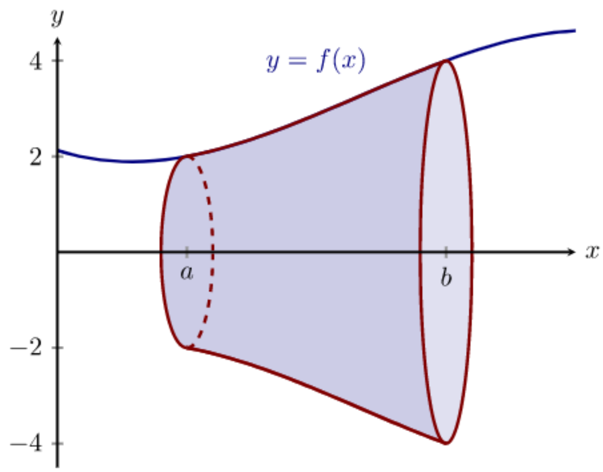
$$x = 9\cos\theta\sin\phi$$
$$y = 9\sin\theta\sin\phi$$
$$z = 9\cos\phi$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

Another example of a parametrization which revisits chapter 12 is that of a plane. A plane passing through a point P with vectors $\vec{a}$ and $\vec{b}$ (nonzero and nonparallel) in the plane is parametrized by

$$\vec{r}(s, t) = \vec{0P} + s\vec{a} + t\vec{b}$$

One more example is surfaces of revolution. To obtain this, we rotate a graph $y = f(x)$ about the x-axis, for example.



A parametrization is:

$$\vec{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle, \qquad 0 \le \theta \le 2\pi$$

Let's revisit another earlier problem: that of finding tangent planes to surfaces. Recall that to find tangent vectors, we needed curves in the surface passing through the desired point, then took derivatives. If the surface is parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

and the point is $P_0 = \vec{r}(u_0, v_0)$, then we can take the two grid curves which pass through P_0 are

$$\begin{aligned} C_u: &\quad \vec{r}(u, v_0) &\quad \text{hold } v \text{ constant at } v_0 \\ C_v: &\quad \vec{r}(u_0, v) &\quad \text{hold } u \text{ constant at } u_0 \end{aligned}$$

Then, two tangent vectors at P_0 are

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$$
$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v}(u_0, v_0)$$

If we wanted a normal vector to S at P_0 we would take $\vec{r}_u \times \vec{r}_v$. As long as $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, we're good.

Definition 1. *We call a parametrization $\vec{r}(u,v)$ of S smooth at P_0 if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.*

So, a parametrization of the tangent plane is

$$T_{P_0}S(s,t) = 0\vec{P}_0 + s\vec{r}_u + t\vec{r}_v \,.$$

Example 4. Find the tangent plane to the surface parametrizeed by $\vec{r}(u,v) = \langle u^2 - v^2, u + v, u^2 + 3v\rangle$ at the point $(3,1,1)$.

Solution:

We begin by finding the u_0 and v_0 such that $\vec{r}(u_0, v_0) = \langle 3, 1, 1 \rangle$:

$$\begin{cases} u^2 - v^2 = 3 \\ u + v = 1 \\ u^2 + 3v = 1 \end{cases}$$

We can now do some algebra to get $u^2 - v^2 = (u+v)(u-v) = (1)(u-v) = 3$ this with the second equation gives $u = 2$ and $v = -1$. We can check that this works with the rest of the equations. So $\vec{r}(2, -1) = \langle 3, 1, 1 \rangle$.

$$\frac{\partial \vec{r}}{\partial u} = \langle 2u, 1, 2u \rangle, \qquad \frac{\partial \vec{r}}{\partial u}(2, -1) = \langle 4, 1, 4 \rangle = \vec{r}_u$$
$$\frac{\partial \vec{r}}{\partial v} = \langle -2v, 1, 3 \rangle, \qquad \frac{\partial \vec{r}}{\partial v}(2, -1) = \langle 2, 1, 3 \rangle = \vec{r}_v$$

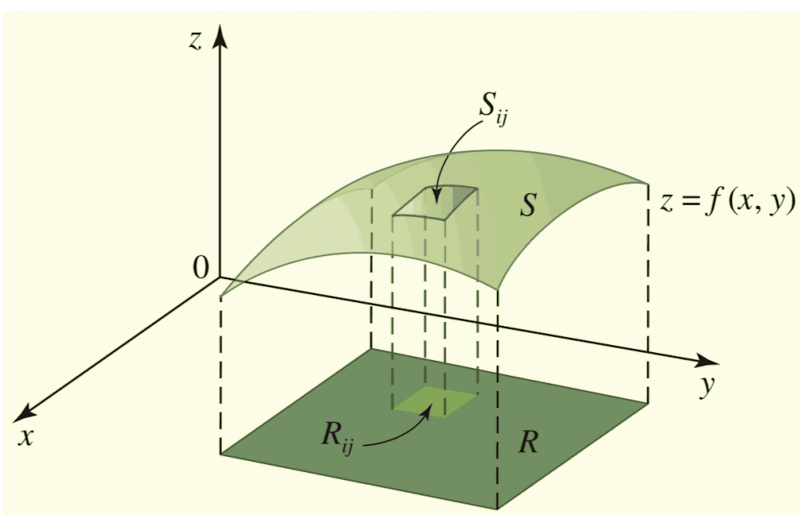
Tangent plane is parametrized by

$$T_{(3,1,1)}S(s, t) = \langle 3, 1, 1 \rangle + s\langle 4, 1, 4 \rangle + t\langle 2, 1, 3 \rangle$$
$$= \langle 3 + 4s + 2t, 1 + s + t, 1 + 4s + 3t \rangle$$

We can now ask about finding surface areas of surfaces. Suppose we have

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

for $(u, v) \in D$. To approximate the area, we'll "tile" it with flat parallelograms, then let them get smaller



$$\Delta S_{ij} = \text{area of parallelogram over } S_{ij}$$
$$= ||\vec{r}_u^* \times \vec{r}_v^*||\Delta u \Delta v$$

This gives the surface area of S as:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta S_{ij} = \int_S dS = \int\int_D ||\vec{r}_u \times \vec{r}_v||\, dA$$

So

$$dS = ||\vec{r}_u \times \vec{r}_v||\, dA$$

Example 5. Find the surface area of the cylinder described by $x^2 + y^2 = 4$, $0 \le z \le 3$

Solution:

Begin by parametrizing the surface. In cylindrical, it's described by $r = 2$, $0 \le z \le 3$. So,

$$\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle, \qquad 0 \le \theta \le 2\pi,\ 0 \le z \le 3$$

$$\vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle, \qquad \vec{r}_z = \langle 0, 0, 1\rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$||\vec{r}_\theta \times \vec{r}_z|| = \sqrt{4\cos^2\theta + 4\sin^2\theta + 0} = \sqrt{4} = 2$$

So

$$A(S) = \int\int_S dS = \int\int_D ||\vec{r}_\theta \times \vec{r}_z||\, dA = \int_0^{2\pi} \int_0^3 2\, dz\, d\theta = 12\pi$$

Extra Problems

1. Find a parameterization for the plane given implicitly by the equation $x + y - 2z = 4$.
2. Find a parameterization for the sphere $(x-1)^2 + (y-2)^2 + (z+3)^2 = 9$.
3. Find the tangent plane to the surface $(u, v) = (u + 2v + 2, u^2 v + v + 1, v^3 + 2u - 1)$ when $u = v = 0$.
4. The vector field $F = \langle P, Q\rangle = \frac{\langle -y, x\rangle}{x^2+y^2}$ is defined on $\mathbb{R}^2$ minus the origin, and satisfies $\frac{Q}{x} - \frac{P}{y} = 0$. Show that

$$\int_C F \cdot dr = 2\pi$$

for any loop (with no self-intersections) travelling counter-clockwise around the origin. (One can interpret $F = \nabla\theta$, where θ is the angle made with the positive x-axis.)