University of Notre Dame Calculus III

LECTURE 31: SURFACE INTEGRALS

Surface Integrals

Definition 1. *The surface integral of f over S (or scalar surface integral) is given by*

$$\int\int_S f\,dS = \int\int_D f(\vec{r}(u,v))\,||\vec{r}_u \times \vec{r}_v||\,dA$$

Example 1. Compute the surface integral $\int\int_S xyz\,dS$ where S is the piece of the cone $z^2 = x^2 + y^2$ in the first octant, below $z = 1$.

Solution:

First, parametrize the surface using cylindrical coordinates:

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle, \qquad 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}$$

$$\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle, \qquad \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$||\vec{r}_r \times \vec{r}_\theta|| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$$

$$\int\int_S xyz\,dS = \int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin\theta(\sqrt{2}r)\,dr\,d\theta = \frac{\sqrt{2}}{5}\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta$$
$$= \frac{\sqrt{2}}{5}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{10}$$

An Application

If the surface S has density $\rho(x,y,z)$, the mass of S is $m = \int\int_S \rho\,dS$ and the center of mass is

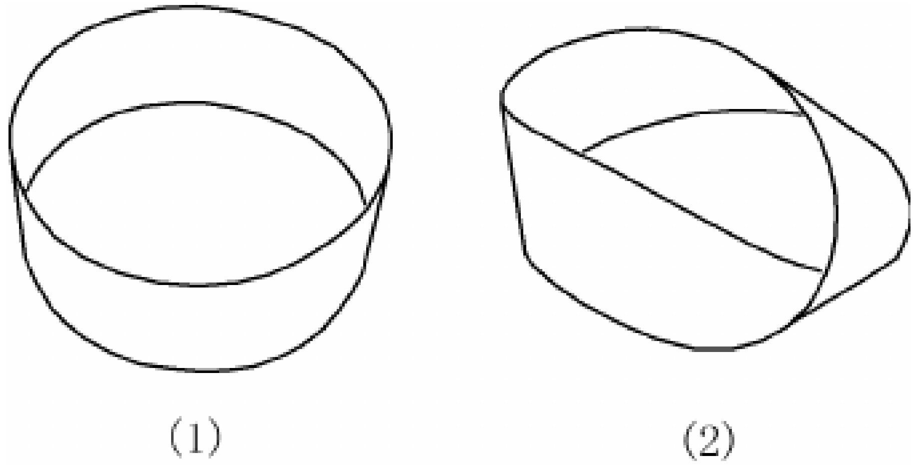
$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{m}\int\int_S x\rho\,dS, \frac{1}{m}\int\int_S y\rho\,dS, \frac{1}{m}\int\int_S z\rho\,dS\right)$$

To continue with the next type of integral (also, the last!) we need to talk about the orientation of a surface.

Definition 2. *An orientation on a surface S is a choice of a continuous unit normal vector field on S.*

Fact 3. *If a surface is orientable, it has exactly two orientations*

Ex:



(1) is orientable, but (2) is not.

If the two orientations are $\vec{n}_1$ and $\vec{n}_2$, then

$$\vec{n}_1 = -\vec{n}_2 \ .$$

If S is parametrized by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} \qquad -\vec{n} = \frac{\vec{r}_v \times \vec{r}_u}{||\vec{r}_v \times \vec{r}_u||}$$

are the two choices of orientation.

Definition 4. *A surface which is the boundary of a solid is called a* <u>*closed surface*</u>*. On a closed surface, the* <u>*positive orientation*</u> *is the outward one.*

Example 2. Find the upward pointing orientation on the surface which is the graph of $f(x,y) = x^2 + y^2$ over $x^2 + y^2 \le 9$.

Solution:

First parametrize the surface

$$\vec{r}(r,\theta) = r\cos\theta, \vec{r}\sin\theta, r^2, \qquad 0 \le r \le 3,\ \ 0 \le \theta \le 2\pi$$

Then $\vec{r}_r = \langle \cos\theta, \sin\theta, 2r \rangle$ $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle$$

$$||\vec{r}_\theta \times \vec{r}_r|| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

Was $\vec{r}_\theta \times \vec{r}_r$ the right order? To check, we check its direction at a point on the surface:
Say we take $(r,\theta) = (1, \frac{\pi}{2})$, $\vec{r}(1, \frac{\pi}{2}) = \langle 0, 1, 1 \rangle$, $\vec{r}_\theta \times \vec{r}_r(1, \frac{\pi}{2}) = \langle 0, 2, -1 \rangle$
This is pointing downward, so we chose the wrong one. To fix this, switch the order, i.e., multiply by -1. So, $\vec{n}$ is

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{||\vec{r}_r \times \vec{r}_\theta||} = \frac{\langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle}{r\sqrt{4r^2+1}}$$