University of Notre Dame Calculus III

LECTURE 31: SURFACE INTEGRALS

Surface Integrals

Definition 1. *The surface integral of f over S (or scalar surface integral) is given by*

$$\int\int_S f\,dS = \int\int_D f(\vec{r}(u,v))\,||\vec{r}_u \times \vec{r}_v||\,dA$$

Example 1. Compute the surface integral $\int\int_S xyz\,dS$ where S is the piece of the cone $z^2 = x^2 + y^2$ in the first octant, below $z = 1$.

Solution:

An Application

If the surface S has density $\rho(x,y,z)$, the mass of S is $m = \int\int_S \rho\,dS$ and the center of mass is

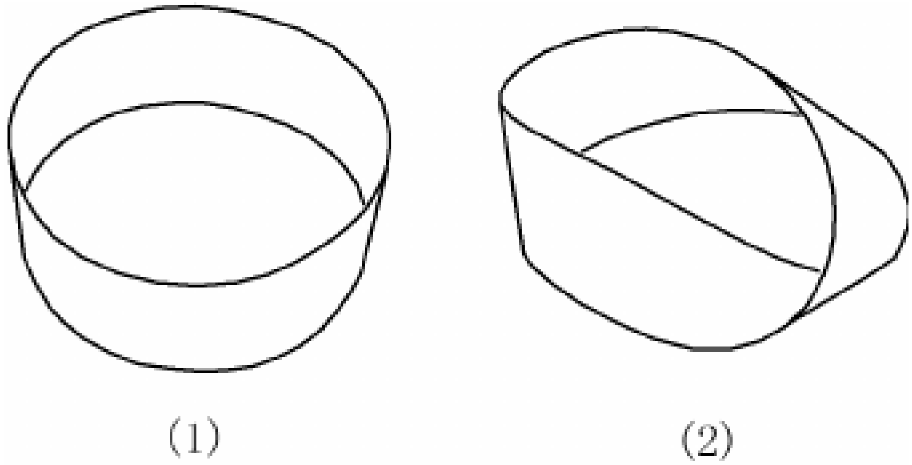
$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{m}\int\int_S x\rho\,dS, \frac{1}{m}\int\int_S y\rho\,dS, \frac{1}{m}\int\int_S z\rho\,dS\right)$$

To continue with the next type of integral (also, the last!) we need to talk about the orientation of a surface.

Definition 2. *An orientation on a surface S is a choice of a continuous unit normal vector field on S.*

Fact 3. *If a surface is orientable, it has exactly two orientations*

Ex:



(1) is orientable, but (2) is not.

If the two orientations are $\vec{n}_1$ and $\vec{n}_2$, then

$$\vec{n}_1 = -\vec{n}_2 \ .$$

If S is parametrized by $\vec{r}(u, v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} \qquad -\vec{n} = \frac{\vec{r}_v \times \vec{r}_u}{||\vec{r}_v \times \vec{r}_u||}$$

are the two choices of orientation.

Definition 4. *A surface which is the boundary of a solid is called a* <u>*closed surface*</u>*. On a closed surface, the* <u>*positive orientation*</u> *is the outward one.*

Example 2. Find the upward pointing orientation on the surface which is the graph of $f(x, y) = x^2 + y^2$ over $x^2 + y^2 \leq 9$.

Solution: