University of Notre Dame Calculus III

LECTURE 32: FLUX INTEGRAL

Flux

The concept of flux is measuring the rate at which something flows through a surface (i.e. air through a butterfly net).

$$d\vec{S} = \vec{n}\,dS = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}\,(||\vec{r}_u \times \vec{r}_v\,dA) = (\vec{r}_u \times \vec{r}_v)\,dA$$

Definition 1. *If $\vec{F}$ is a continuous vector field defined on a surface S which has orientation $\vec{n}$, then the flux of $\vec{F}$ across S is*

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_S \left(\vec{F}\cdot\vec{n}\right)\,dS = \int\int_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

Where S is parametrized by $\vec{r}(u,v)$ and $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$.

Example 1. Find the flux of $\vec{F} = \langle x, y, z\rangle$ across the helicoid parametrized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$ with upward orientation.

Solution:

Start by finding the correct orientation

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle, \qquad \vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -s\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\rangle$$

Since the $\hat{k}$-component is positvie, we have the right order. Since $||\vec{r}_u \times \vec{r}_v||$ concells out in the flux integral there's no need to compute it.

$$\vec{F}(\vec{r}(u,v)) = \langle u\cos v, u\sin v, v\rangle$$

$$\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v) = u\cos v\sin v - u\sin v\cos v + uv = uv$$

So, the flux is

$$\int\int_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^{\frac{\pi}{2}} uv\,dv\,du = \int_0^1 \frac{1}{2}uv^2\Big|_0^{\frac{\pi}{2}}\,du = \int_0^1 \frac{\pi^2}{8}u\,du$$
$$= \frac{\pi^2}{16}$$

Example 2. Compute $\int\int_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \vec{G}$. $\vec{G} = \langle -2yz, y, 3x\rangle$, and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ with the upward orientation.

Solution:

Begin by parametrizing S. Using cylindrical, the paraboloid is $z = 5 - r^2$, so

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 5 - r^2$$

where $0 \le \theta \le 2\pi$ and $0 \le r \le ?$ To get this upper bound on r, we need to know the intersection of the paraboloid with $z = 1$.

$$1 = z = 5 - x^2 - r^2 \Longleftrightarrow x^2 + y^2 = 4 \Longleftrightarrow r = 2$$

So a parametrization is

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 5 - r^2\rangle$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Now $\vec{r}_r = \langle \cos\theta, \sin\theta, -2r\rangle$ and $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\sin\theta & 0 \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle$$

Since the $\hat{k}$-component is positive, this was the correct order. Now

$$\vec{F} = \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} = \langle 0, -2y - 3, 2z\rangle$$

So

$$\vec{F}(\vec{r}(r,\theta)) = \langle 0, -2r\sin\theta - 3, 10 - 2r^2\rangle$$

and

$$\vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) = -4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3$$

Finally

$$\begin{aligned} \int\int_S \vec{F}\cdot d\vec{S} &= \int_0^{2\pi}\int_0^2 \left(-4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3\right)\, dr\, d\theta \\ &= \int_0^{2\pi} \left(-r^4\sin^2\theta - 2r^3\sin\theta + 5r^2 - \frac{1}{2}r^4\right)\Bigg|_0^2 d\theta \\ &= \int_0^{2\pi} (-16\sin^2\theta - 16\sin\theta + 12)\, d\, d\theta \\ &= \int_0^{2\pi} (8\cos 2\theta - 16\sin\theta + 4)\, d\theta \\ &= (4\sin 2\theta + 16\cos\theta + 4\theta)\Big|_0^{2\pi} = 8\pi \end{aligned}$$