University of Notre Dame Calculus III

Stokes' Theorem

Let S be a surface with boundary C and orientation $\vec{n}$:
We say that C has orientation consistent with S if while traversing C with your head in the direction of $\vec{n}$, the surface is on your left. In this sense, we say S induces an orientation on C, and C with this orientation is denoted ∂S.

Theorem 1. *Stokes' Theorem Let S be an oriented, piecewise-smooth surface, which is bounded by a simple, closed, piecewise-smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is C^1 in an open region containing S. Then*

$$\int_C \vec{F}\cdot d\vec{r} = \int\int_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$

Let's revisit the example from earlier

Example 1. Compute $\int\int_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \operatorname{curl}\vec{G}$. $\vec{G} = \langle -2yz, y, 3x\rangle$, and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ with the upward orientation.

Solution:

We can apply Stokes theorem to this case. C is the circle $x^2 + y^2 = 4$, $z = 1$. The orientation we want on C is the counterclockwise one when viewed from above. So, a parametrization of C is

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle$$

for $0 \le t \le 2\pi$. By Stokes'

$$\int\int_S \operatorname{curl}\vec{G}\cdot d\vec{S} = \int_C \vec{G}\cdot d\vec{r}$$

So

$$\vec{G}(\vec{r}(t)) = \langle -4\sin t, 2\sin t, 6\cos t\rangle$$
$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$\vec{G}(\vec{r}(t))\cdot\vec{r}'(t) = 8\sin^2 t + 4\sin t\cos t$$

$$\begin{aligned}\int_C \vec{G}\cdot d\vec{r} &= \int_0^{2\pi}\left(8\sin^2 t + 4\sin t\cos t\right)dt\\ &= \int_0^{2\pi}(4 - 4\cos 2t + 4\sin t\cos t)\,dt\\ &= (4t - \sin t + 2\sin^2 t)\Big|_0^{2\pi} = 8\pi\end{aligned}$$

This is much quicker!

Example 2. Compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle xy, 2z, 3y\rangle$ and C is the curve of intersection between $x+z=5$

and $x^2 + y^2 = 9$, oriented counterclockwise when viewed from above.

Solution:

Parametrizing C wouldn't be too bad

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, 5 - 3\cos t \rangle$$

for $0 \le t \le 2\pi$ but

$$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 3\sin t \rangle \qquad \vec{F}(\vec{r}(t)) = \langle 9\cos t \sin t, 10 - 6\cos t, 9\sin t \rangle$$

So

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -27\cos t \sin^2 t + 30\cos t - 18\cos^2 t + 27\sin^2 t$$

which is awful...

But, we can still use Stokes' theorem! We need a surface with boundary on C. We may as well use the piece of the plane $x + z = 5$ inside the cylinder $x^2 + y^2 = 9$. So

$$\vec{r}(x, y) = \langle x, y, 5 - x \rangle \qquad D = \{x^2 + y^2 \le 9\}$$

In order for S to induce the correct orientation on C, it needs the upward orientation. So

$$\vec{r}_x = \langle 1, 0, -1 \rangle \qquad \vec{r}_y = \langle 0, 1, 0 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1, 0, 1 \rangle$$

which has positive $\hat{k}$-component, so $\vec{r}_x \times \vec{r}_y$ is the correct order

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 3y \end{vmatrix} = \langle 1, 0, -x \rangle$$

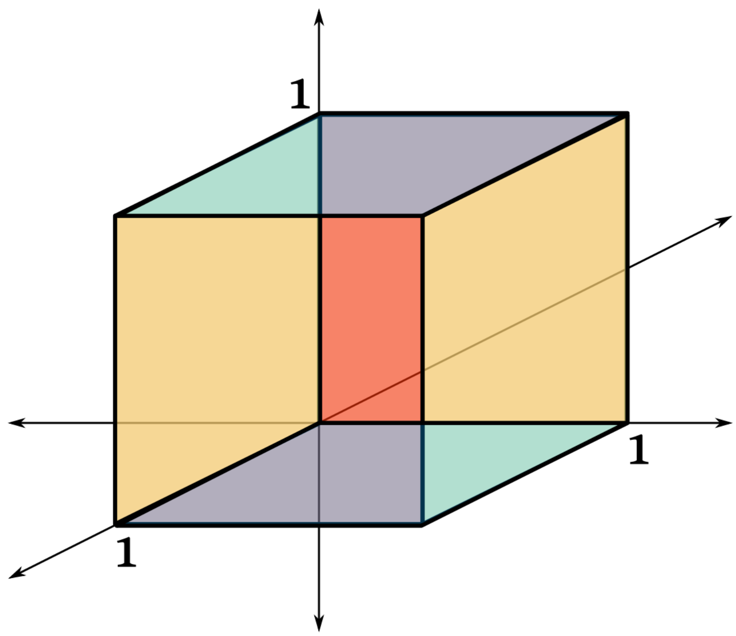
Thus

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int\int_D \langle 1, 0, -x \rangle \cdot \langle 1, 0, 1 \rangle \, dA \\
&= \int\int_D (1 - x)\, dA = \int_0^{2\pi} \int_0^3 (1 - r\cos\theta) r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^3 (r - r^2 \cos\theta)\, dr \, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3}\cos\theta \right) \Big|_0^3 d\theta \\
&= \int_0^{2\pi} \left(\frac{9}{2} - 9\cos\theta \right) d\theta = 9\pi
\end{aligned}$$

Let's close with one more example: a trick. Suppose S_1 and S_2 have the same boundary C, and they both induce the same orientation on C. If everything in question satisfies Stokes' theorem, then

$$\int\int_{S_1} (\text{curl}\,\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int\int_{S_2} (\text{curl}\,\vec{F}) \cdot d\vec{S}$$

I like to call this process "surface swapping".
Suppose we were asked to compute $\int\int_{B_1} (\text{curl}\,\vec{F}) \cdot d\vec{S}$, where B_1 is the surface of the box $[0,1] \times [0,1] \times [0,1]$, with no bottom, where B_1 has the "outward" orientation.



This would require computing 5 surface integrals... quite frustrating, and switching to an integral over the boundary C still requires 4 line integrals... However, using surface swapping, we can replace B_1 by the bottom of the box, B_2, with upward orientation $B_2 = [0,1] \times [0,1] \times \{0\}$, which is much easier!

Extra Problems

1. Use Stokes' theorem to compute $\int_C \mathbf{F} \cdot \mathbf{dr}$ where $= \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, oriented counterclockwise when viewed from above.

2. Use Stokes' theorem to evaluate $\iint_S curl(\mathbf{F}) \cdot \mathbf{dS}$ where $= \langle x^2 \sin z, y^2, xy \rangle$ and S is the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane, having upwards orientation.