University of Notre Dame Calculus III

LECTURE 33: STOKES' THEOREM

Stokes' Theorem

Let S be a surface with boundary C and orientation $\vec{n}$:
We say that C has orientation consistent with S if while traversing C with your head in the direction of $\vec{n}$, the surface is on your left. In this sense, we say S induces an orientation on C, and C with this orientation is denoted ∂S.

Theorem 1. *Stokes' Theorem Let S be an oriented, piecewise-smooth surface, which is bounded by a simple, closed, piecewise-smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is C^1 in an open region containing S. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$$

Let's revisit the example from earlier

Example 1. Compute $\int\int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \text{curl}\,\vec{G}$. $\vec{G} = \langle -2yz, y, 3x\rangle$, and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ with the upward orientation.

Solution:

This is much quicker!

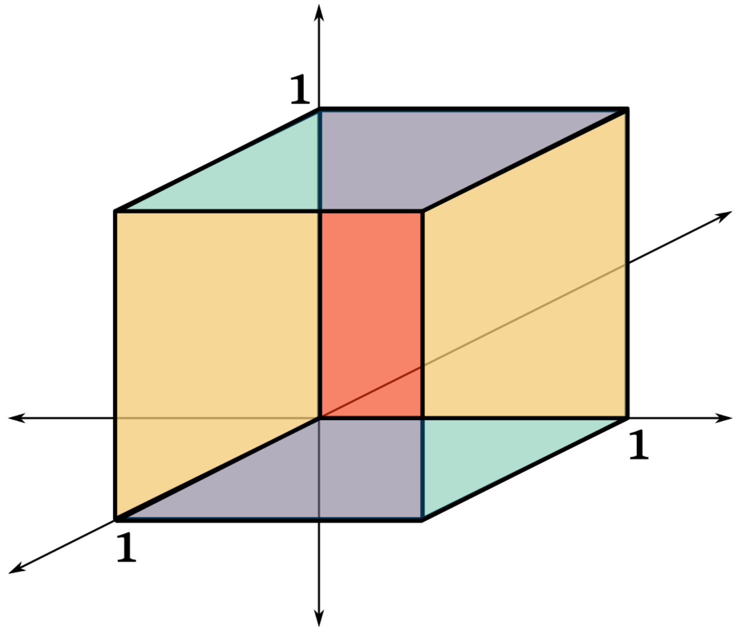
Example 2. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, 2z, 3y\rangle$ and C is the curve of intersection between $x + z = 5$ and $x^2 + y^2 = 9$, oriented counterclockwise when viewed from above.

Solution:

Let's close with one more example: a trick. Suppose S_1 and S_2 have the same boundary C, and they both induce the same orientation on C. If everything in question satisfies Stokes' theorem, then

$$\int\int_{S_1} (\text{curl}\,\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int\int_{S_2} (\text{curl}\,\vec{F}) \cdot d\vec{S}$$

I like to call this process "surface swapping".
Suppose we were asked to compute $\int\int_{B_1} (\text{curl}\,\vec{F}) \cdot d\vec{S}$, where B_1 is the surface of the box $[0,1] \times [0,1] \times [0,1]$, with no bottom, where B_1 has the "outward" orientation.



This would require computing 5 surface integrals... quite frustrating, and switching to an integral over the boundary C still requires 4 line integrals... However, using surface swapping, we can replace B_1 by the bottom of the box, B_2, with upward orientation $B_2 = [0,1] \times [0,1] \times \{0\}$, which is much easier!

Extra Problems

1. Use Stokes' theorem to compute $\int_C \mathbf{F} \cdot \mathbf{dr}$ where $= \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, oriented counterclockwise when viewed from above.

2. Use Stokes' theorem to evaluate $\iint_S curl(\mathbf{F}) \cdot \mathbf{dS}$ where $= \langle x^2 \sin z, y^2, xy \rangle$ and S is the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane, having upwards orientation.