

University of Notre Dame Calculus III

LECTURE 34: THE DIVERGENCE THEOREM

The Divergence Theorem

Theorem 1. Let E be a bounded solid region in \mathbb{R}^3 with boundary S , where S consists of finitely many piecewise smooth, closed, orientable surfaces, each of which oriented with normals pointing away from E . Let \vec{F} be a vector field which is C^1 on an open region containing E . Then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_E (\operatorname{div} \vec{F}) dV$$

Example 1. Compute the flux of $\vec{F} = \langle z, y, x \rangle$ across the sphere $x^2 + y^2 + z^2 = 7921$, where the sphere is oriented outward.

Solution:

$\operatorname{div} \vec{F} = 0 + 1 + 0$. This sphere bounds the ball $B : x^2 + y^2 + z^2 \leq 7921 = 89^2$. By the divergence theorem

$$\begin{aligned} \int \int_S \vec{F} \cdot d\vec{S} &= \int \int \int_B (\operatorname{div} \vec{F}) dV = \int \int \int_B 1 dV = \operatorname{Vol}(B) = \frac{4}{3}\pi(89)^3 \\ &= \frac{4}{3}\pi(704969) \end{aligned}$$

Example 2. Compute $\int \int_S \vec{F} \cdot d\vec{S}$ where S is the boundary of the solid bounded by $z = 4 - x^2 - y^2$ and the xy -plane, oriented positively, where $\vec{F} = \langle x^2, xy, z \rangle$.

Solution:

$$\operatorname{div} \vec{F} = 2x + 2 + 1 = 3x + 1$$

$$\begin{aligned} \int \int_S \vec{F} \cdot d\vec{S} &= \int \int \int_E (\operatorname{div} \vec{F}) dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r \cos \theta + 1)r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r^2 \cos \theta + r) dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 (3r^2 \cos \theta + r)(4 - r^2) dr d\theta \\ &= \int_0^{2\pi} \int_0^2 (12r^2 \cos \theta + 4r - 3r^4 \cos \theta - r^3) dr d\theta \\ &= \int_0^{2\pi} \left[4r^3 \cos \theta + 2r^2 - \frac{3}{5}r^5 \cos \theta - \frac{1}{4}r^4 \right] \Big|_0^2 d\theta \\ &= \int_0^{2\pi} \left(32 \cos \theta + 8 - \frac{96}{5} \cos \theta - 4 \right) d\theta \\ &= \int_0^{2\pi} \left(\frac{64}{5} \cos \theta + 4 \right) d\theta \\ &= 8\pi \end{aligned}$$

The divergence theorem can also help in computing surface integrals over non-closed surfaces. This is done by "capping off" the surface, using the divergence theorem, then subtracting off the surface integral over the "cap". Formally:

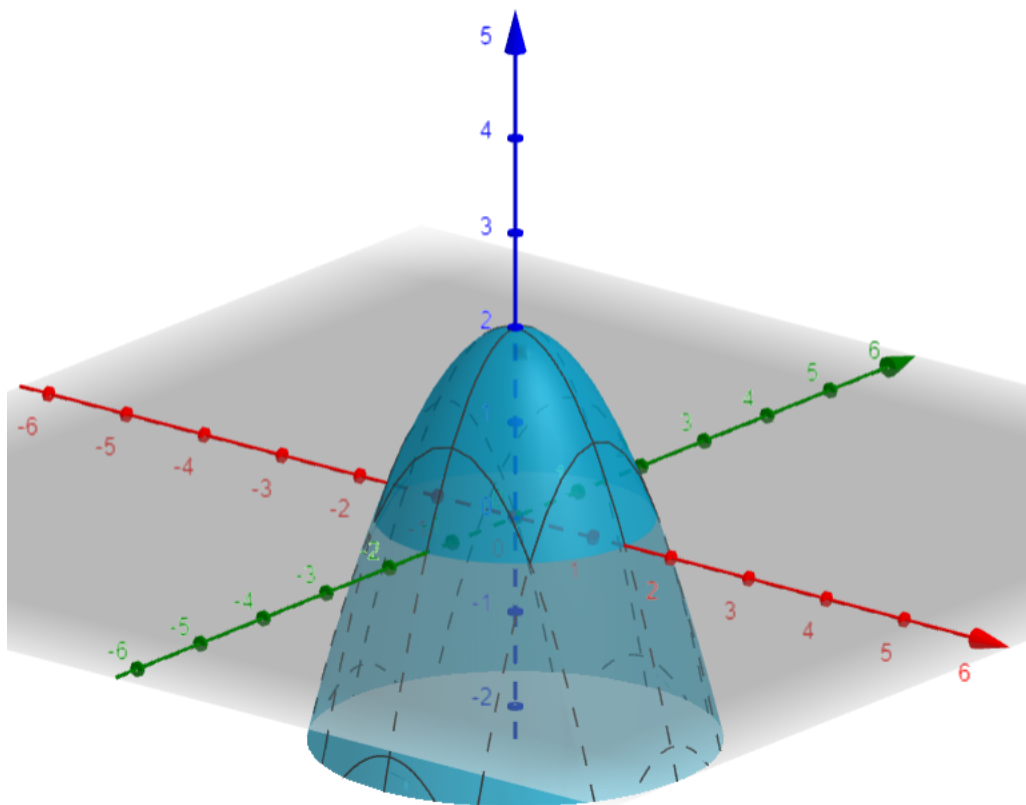
If S_1 is a surface which isn't closed, but $S_1 \cup S_2$ is for some surface S_2 , and $S_1 \cup S_2$ bounds a solid E and is oriented outward then

$$\int \int_{S_1} \vec{F} \cdot d\vec{S} + \int \int_{S_2} \vec{F} \cdot d\vec{S} = \int \int_{S_1 \cup S_2} \vec{F} \cdot d\vec{S} = \int \int \int_E (\operatorname{div} \vec{F}) dV$$

Implies

$$\int \int_{S_1} \vec{F} \cdot d\vec{S} = \int \int \int_E (\operatorname{div} \vec{F}) dV - \int \int_{S_2} \vec{F} \cdot d\vec{S}$$

Example 3. Let $\vec{F} = \langle x + z \arctan(y^3), z^3 \ln(x^2 + 1), 0 \rangle$. Compute the flux of \vec{F} across the surface which is the part of the paraboloid $x^2 + y^2 + z = 2$ above plane $z = 1$, oriented upward.



$$z = 2 - x^2 - y^2$$

Solution:

To cap this surface off, we fill in the hole at the bottom with the surface S_2 given by $x^2 + y^2 \leq 1, z = 1$. The correct orientation to give S_2 so that $S_1 \cup S_2$ has a continuous normal vector field is the downward orientation. Then

$$\int \int_{S_1} \vec{F} \cdot d\vec{S} = \int \int \int_E (\operatorname{div} \vec{F}) dV - \int \int_{S_2} \vec{F} \cdot d\vec{S}$$

where E is the solid bounded by $S_1 \cup S_2$.

$$\operatorname{div} \vec{F} = 1 + 0 + 0 = 1$$

So

$$\begin{aligned} \int \int \int_E (\operatorname{div} \vec{F}) dV &= \int_0^{2\pi} \int_0^1 \int_0^{2-r^2} r dz dr d\theta = \int_0^{2\pi} \int_0^1 r z \Big|_0^{2-r^2} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r - r^3 - r) dr d\theta = \int_0^{2\pi} \int_0^1 (r - r^3) dr d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \right) \Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2} \end{aligned}$$

Now, we compute $\int \int_{S_2} \vec{F} \cdot d\vec{S}$. Parametrize S_2

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 1 \rangle, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

Find the downward normals for S_2

$$\vec{r}_r = \langle \cos \theta, \sin \theta, 0 \rangle \quad \vec{r}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

The downward normals are given by

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \end{vmatrix} = \langle 0, 0, -r \rangle$$

$$\vec{F}(\vec{r}(r, \theta)) = \langle r \cos \theta + \arctan(r^3 \sin^3 \theta), \ln(r^2 \cos^2 \theta + 1), 0 \rangle$$

$$\vec{F}(\vec{r}(r, \theta)) \cdot (\vec{r}_\theta \times \vec{r}_r) = 0$$

So, $\int \int_{S_2} \vec{F} \cdot d\vec{S} = 0$. Thus

$$\int \int_{S_1} \vec{F} \cdot d\vec{S} = \int \int \int_E (\operatorname{div} \vec{F}) dV - \int \int_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Extra Problems

1. Use the divergence theorem to find the flux of $\mathbf{F} = \langle xz, yz, xy \rangle$ through the surface S comprising of the piece of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$, oriented with the upwards unit normal.
2. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 2y + 3$, oriented counterclockwise when viewed from above. Use Stokes' theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle 2e^y - z, \cos(yz), xe^y \rangle$.
3. Find the outward flux of the vector field $\mathbf{F} = \langle z, y, x \rangle$ over the unit sphere centered at the origin.
4. Use the divergence theorem to find the outward flux of $\mathbf{F} = \langle 3x, xy, 2xz \rangle$ through E , where E is cube bounded by $x = 0, 1, y = 0, 1, z = 0, 1$.
5. The electric field associated to a charge distribution ρ satisfies Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ for some constant ϵ_0 depending on the medium. In a large region of space $\mathbf{E} = \langle x^3, zx, y^2 \rangle$. Find the total charge inside the cylindrical region $\{-1 \leq z \leq 1, x^2 + y^2 \leq 1\}$. (Here \mathbf{E} has units newtons per coulomb, and ρ has units coulombs per cubic meter).
6. An infinite plate coinciding with the xy -plane has charge density σ (units: coulombs per square meter). In particular this means that if U is a vertical cylinder $\{-1 \leq z \leq 1, x^2 + y^2 \leq R\}$, then the total charge in U is $\pi R^2 \sigma$ coulombs. Use Gauss's relation and a symmetry argument to find the associated electric field outside this plate.
7. Let $\mathbf{F} = \langle x^3, y^3 \rangle$ be a vector field in the plane, and $D \subset \mathbb{R}^2$ the domain inside the ellipse $x^2 + y^2/4 = 1$. Find the flux of \mathbf{F} out of D , i.e. $\int_D \mathbf{F} \cdot \mathbf{n} \, ds$ where \mathbf{n} is the outward unit normal of D .